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# Another Proof on generalized Viviani's Theorem

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**Abstract**

We will give a proof, from the perspective of rotating axes, of the generalized Viviani's theorem: In the case of a regular polygon of $n$ sides, the sum of signed perpendiculars from an arbitrary point on the plane equals $n$ times the apothem.

**AMS subject classification:**
**Keywords:**

## 1. Introduction

Viviani's theorem [3], proposed by Fermat, solved by Torricelli and published by Viviani, states that the sum of the distances from any interior point to the sides of an equilateral triangle equals the length of the triangle's altitude.

The proof is simple. We break the equilateral triangle into three smaller triangles and calculate their areas. The idea of proof can be extended to arbitary points on a plane, not just the interior points, if we consider a perpendular is negative if it is completely outside the equilateral triangle. Therefore we have the generalizaton: the sum of signed perpendiculars from an arbitary interior point of a equilateral triangle equals the length of the triangle's altitude [2].

It is not hard to see that the proof works on a regular $n$-gon as well:

**Theorem 1.1.** In the case of a regular polygon of $n$ sides, the sum of signed perpendiculars from an arbitrary point on the plane equals $n$ times the apothem (i.e. the distance from the center to the midpoint of a side).

In this article, we are going to provide an alternative proof from the perspective of rotating axes. It is worthnoting that recently Zhou [4] have extended the result to general polyhedrons in all dimensions, he gives a necessary and sufficient conditions for the sum of signed perpendiculars to be a constant. This is his answer to an problem posed by Abboud [1] and his argument uses only dot product of vectors.

## 2. Our proof

We state a very simple fact:

**Lemma 2.1.** Let the coordinates of a point on $\mathbf{R}^2$ be $(x_0, y_0)$. If the axes are rotated by an angle $\theta$ anticlockwisely, then the new coordinates become $(x_\theta, y_\theta) = (x\cos\theta + y\sin\theta, y\cos\theta - y\sin\theta)$.

Using the notation in the lemma, we have

$$\sum_{j=0}^{n-1}(x_{\frac{j\pi}{n}}, y_{\frac{j\pi}{n}}) = 0.$$

The equality can be proved by compound angle formula or by identifying $(x, y)$ as $x + \sqrt{-1}y$.

We are ready to proof Theorem 1.1.

*Proof.* Let $O$ and $l$ be the center and a side of a regular polygon. Adjust the axes such that $O$ is the origin and the the equation of $l$ becomes $y = -h$, where $h$ is the apothem. For an arbitrary point $P$ with coordinates $(x_0, y_0)$, the signed perpendicular is $y_0 + h$, which is negative if $P$ is below $l$.

The sum of signed perpendicular is therefore

$$\sum_{j=0}^{n-1}(y_{\frac{j\pi}{n}} + h) = nh.$$

■

## 3. A corollary

We have the following corollary.

**Corollary 3.1.** Let $P$ be an interior point of a regular $2n$-gon. Slice the polygon into $2n$ triangles and divide them into two groups of non-adjacent triangles. The two sums of areas of triangles are equal.

For square, the result is trivial. Lets consider $n > 3$. In [2], Samson proves the case of regular hexagon and we follows his proof.

*Proof.* Let $l_0, \ldots, l_{2n-1}$ be the $2n$ sides of the regular $2n$-gon. Extend $l_0, l_3, \ldots, l_{2n-2}$, we get a new regular $n$-gon. Extend $l_1, l_3, \ldots, l_{2n-1}$, we get a different but equivalent regular $n$-gon. Hence by Theorem 1.1, the sum of perpendiculars to $l_0, l_3, \ldots, l_{2n-2}$ equals the sum of perpendiculars to $l_1, l_3, \ldots, l_{2n-1}$. The result follows. ■

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## References

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