Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 6 (2015), pp. 4501–4510

http://www.ripublication.com/gjpam.htm

The uniform convergence of Schwarz method for quasi-variationnal inequalities

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Abstract

In this paper, we prove a maximum norm analysis of an overlapping Schwarz method on nonmateching girds for a quasi-variational inequality, where the obstacle and the second member depends of the solution, our result improves and generalizes some previous results.

AMS subject classification: 05C38, 15A15, 05A15, 15A18.
Keywords: Schwarz method, quasi-variational inequalities, L^∞-error estimates.

1. Introduction

Schwarz method-who's depends on dismantling of the space has known Rapidly developed in the last two twenties, because of the tremendous development in the computer world.

This method enables us to abridge a problem of considerable size to the issue of small size and low cost in terms of the solution, and also dismantling of the total domain irregular to a sub-domains regular and simple.

Schawrz method is able to simplify complex issues on engineering through disassembly and help us count parallel on profit time and cost in addition to the properties of convergence. This method has been used in many works in regards to of elliptic linear problems. Ether in inequations variational and quasi- variational just few problems have done.

The Schwarz method is interesting for the approximation of the inequations variational and quasi-variational by using the finite element.

In this paper, we will study an inequality quasi-variational where the obstacle and the second member is depended to the solution, our results generalize and extend some previous results as in [8], where the authors obtained the following approximation:

$$\|u_1 - u_{1h}^{n+1}\|_\infty \le Ch^2|\log h|^3, i = 1, 2$$

for the problem:

$$\begin{cases} a(u, v-u) \ge (f, v-u) & \text{in } \Omega, \forall v \in K \\ u \le \psi; & v \le \psi \end{cases}$$

and we find in [11] the same result for the following problem:

$$\begin{cases} a(u, v-u) \ge (f(u), v-u) & \text{in } \Omega, \forall v \in K(u) \\ u \le \psi; & v \le \psi \end{cases}$$

As for the noncoercive variational inequality has been reached in [18] the same approximation above mentioned.

On the other hand done in [17] study a quasi-variational inequality related control egordic problem:

$$\begin{cases} b(u_\alpha, v-u_\alpha) \ge (f + ru_\alpha, v-u_\alpha) & \text{in } \alpha \in (0, 1) \\ u_\alpha \le Mu_\alpha; & v \le Mu_\alpha \end{cases}$$

and we get the following result:

$$\|u_{\alpha_i} - u_{\alpha_i h}^{n+1}\|_\infty \le C\alpha^{-2}h^2|\log h|^4, \quad i = 1, 2.$$

Finally in [10] the authors studied the following problem:

$$\begin{cases} a(u, v-u) \ge (f, v-u) & \forall v \in K \\ u \le Mu; & Mu \ge 0 \\ Mu = k + \inf\limits_{\varepsilon \ge 0, x+\varepsilon \in \overline{\Omega}} u(x+\varepsilon); & \\ \dfrac{\partial u}{\partial \eta} = \varphi; & \text{in } \Gamma_0 \text{ and } u = 0 \text{ in } \Gamma/\Gamma_0 \end{cases}$$

and they obtained the result:

$$\|u_1 - u_{1h}^{n+1}\|_\infty \le Ch^2|\log h|^3, i = 1, 2$$

for our work we claim about the general problem where the second member and the obstacle related to the solution:

$$\begin{cases} a(u, v-u) \geq (f(u), v-u) & \text{in } \Omega, \forall v \in K_g(u) \\ u \leq Mu; & v \leq Mu \\ u = g; & \text{on } \partial\Omega \end{cases}$$

We will improve and generalize previous results obtained in [5],[10],[11],[18] and [17], where we get the following approximation:

$$\|u_1 - u_{1h}^{n+1}\|_\infty \leq Ch^2 |\log h|^2, i = 1, 2$$

2. The continuous problem

2.1. Notations and assumptions

Let Ω be an open in $\mathbb{R}^n$, with sufficiently smooth boundary $\partial\Omega$.For $u, v \in H^1(\Omega)$, consider the bilinear form as follows:

$$a(u, v) = \int_\Omega \Big(\sum_{1\leq i,j\leq n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{1\leq i,j\leq n} ai(x) \frac{\partial u}{\partial x_i} + a_0(x) u.v \Big) dx, \tag{2.1}$$

where $a_{ij}(x), a_i(x), a_0(x), x \in \overline{\Omega}, 1 \leq i, j \leq n$ are sufficiently smooth coefficients and satisfy the following conditions:

$$\sum_{1\leq i,j\leq n} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2, \xi \in \mathbb{R}^n, \nu > 0 \tag{2.2}$$

$$a_0(x) \geq \beta > 0, \tag{2.3}$$

where β is a constant. M is operator given by

$$Mu = k + \inf_{\varepsilon \geq 0, x+\varepsilon \in \overline{\Omega}} u(x+\varepsilon),$$

where $k > 0$ and M satisfies

$$Mu \in W^{2,\infty}(\Omega), Mu \geq 0, \text{on}\partial\Omega : 0 \leq g \leq Mu \tag{2.4}$$

where g is a regular function defined on $\partial\Omega$. Let f be a Lipschitzian non decreasing nonlinear function with rate α satisfying

$$\frac{\alpha}{\beta} < 1, f \in L^\infty(\Omega) \tag{2.5}$$

$K_g(u)$ is an implicit convex and non empty set which defined as follows:

$$K_g = \{v \in H^1(\Omega), v = g \text{on} \partial\Omega, v \leq Mu \text{in} \Omega\}.$$

2.2. Elliptic quasi-variational inequalities

we consider the following problem: Find $u \in k_g(u)$ the solution of

$$\begin{cases} a(u, v-u) \geq (f(u), v-u) & \text{in } \Omega, \forall v \in K_g(u) \\ u \leq Mu; & v \leq Mu \\ u = g; & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

We will present the existence and uniqueness for the solution and an important proposition.

Theorem 2.1. [12] Under the previous conditions thee problem (3.1) has an unique solution $u \in K_g(u)$.More over we have

$$u \in W^{2,p}(\Omega), 2 \leq p \leq \infty.$$

Proposition 2.2. [5] Note that $u = \sigma(f(u)), \tilde{u} = \sigma(f(\tilde{u}))$, for all u and $\tilde{u} \in K_g(u)$ we have

1. if $f(u) \geq f(\tilde{u})$, then $u \geq \tilde{u}$
2. $$\|\sigma(f(u)) - \sigma(f(\tilde{u}))\|_{L^\infty(\Omega)} \leq \frac{1}{\beta}\|f(u) - f(\tilde{u})\|_{L^\infty(\Omega)}.$$

3. The discrete problem

we denote by V_h the standard piecewise linear finite element space, we considered the discrete quasi-variational inequality: Find $u_h \in K_{gh}(u_h)$ such that

$$\begin{cases} a(u_h, v-u_h) \geq (f(u_h), v-u_h) & \forall u_h, v \in K_{gh}(u_h) \\ u_h \leq r_h M u_h & \\ u_h = \pi_h g & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

where $f \in L^\infty(\Omega)$; $Mu_h = k + \inf_{\varepsilon \geq 0, x+\varepsilon \in \overline{\Omega}} u_h(x+\varepsilon)$ and

$$K_{gh}(u_{u_h}) = \{v \in V_h : v = \pi_h g \ \ on \ \ \partial\Omega; \, v \leq r_h M u_h \ \ in \ \ \Omega\}$$

π_h denote the interpolation operator on $\partial\Omega$ and r_h is the usual finite element restriction operator on Ω.

3.1. The discrete maximum principle

We assume that the respective matrices resulting from the discritization of problems (3.1) are M-matrix [20].

Theorem 3.1. [5] Let u and u_h be the solutions of problem (3.1) and (4.1) respectively, there exists a constant C_1 independent of h such that:

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C_1 h^2 \log|h|^2.$$

Similarly, for the continuous case we have:

Proposition 3.2. For all u_h and $\tilde{u_h} \in K_g(u_h)$ we have

1. if $f(u_h) \leq f(\tilde{u_h})$, then $u_h \leq \tilde{u_h}$
2. $$\|\sigma_h(f(u_h)) - \sigma_h(f(\tilde{u_h}))\|_{L^\infty(\Omega)} \leq \frac{1}{\beta}\|f(u_h) - f(\tilde{u_h})\|_{L^\infty(\Omega)}.$$

3.2. Domain decomposition method

We decompose Ω in two sub-domains Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$ and u satisfies the local regularity condition

$$u/_{\Omega_i} \in W^{2,p}(\Omega_i, i = 1, 2 \text{and} 2 \leq p < \infty,$$

denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_1 = \partial\Omega_1 \cap \Omega_2$, $\Gamma_2 = \partial\Omega_2 \cap \Omega_1$, $\Gamma_1 \cap \Gamma_2 = \emptyset$.

3.3. The discrete Schwarz sequences

For $i = 1, 2$ let $V_{h_i} = V_h(\Omega_i)$ be the space of continuous picewise linear function on τ_{h_i}, which vanish on $\partial\Omega \cap \partial\Omega_i$. For $w \in C(\Gamma_i)$, we define

$$V_{h_i}^{(w)} = \{v \in V_{h_i}; v = 0 \quad on \quad \partial\Omega_i \cap \partial\Omega; v = \pi_{h_i}(w) \quad on \quad \Gamma_2\},$$

where τ_{h_i} be a standard regular finite element triangulation in Ω_i, h_i being the mesh size.

We suppose that the two triangulation are mutually independent on Ω_1, Ω_2 a triangle belonging to one triangulation does not necessarily belong to the other. Choosing $u_{1h}^0 = u_{2h}^0 = r_h M u_h$, now we define the discrete counterparts of the continuous Schwarz sequences defined in (3.2) and (3.3) respectively by: $(u_{1h}^{n+1}) \in V_{h_1}^{(u_{2h}^n)}$ where (u_{1h}^{n+1}) is the solution of:

$$\begin{cases} a_1(u_{1h}^{n+1}, v - u_{1h}^{n+1}) \geq (f_1(u_{1h}^n), v - u_{1h}^{n+1}) & \forall v \in V_{h_1}^{(u_{2h}^n)} \\ u_{1h}^{n+1} \leq r_h M u_{1h}^n & v \leq r_h m u_{1h}^n \\ u_{1h}^{n+1} = u_{2h}^n & \text{on } \Gamma_1, v = u_{2h}^n \text{ on } \Gamma_1 \end{cases} \tag{3.2}$$

and $(u_{2h}^{n+1}) \in V_{h2}^{(u_{1h}^{n+1})}$ such that (u_{2h}^{n+1}) is the solution of:

$$\begin{cases} a_2(u_{2h}^{n+1}, v - u_{2h}^{n+1}) \geq (f_2(u_{2h}^n), v - u_{2h}^{n+1}) & \forall v \in V_{h_2}^{(u_{1h}^n)} \\ u_{2h}^{n+1} \leq r_h M u_{2h}^n & v \leq r_h M u_{2h}^n \\ u_{2h}^{n+1} = u_{1h}^n & \text{on } \Gamma_2, v = u_{2h}^n \text{ on } \Gamma_2 \end{cases} \tag{3.3}$$

We will finish this section by the main result which concerning the geometrical convergence of discrete sequences.

Lemma 3.3. Under the conditions in (2.1) to (2.5) the sequences (u_{1h}^{n+1}) and (u_{2h}^{n+1}) : $n \geq 0$ converge geometrically to the unique solution u_h of the discrete problem (3.1), more precisely, there exists a constant $k \in (0, 1)$ such that for all $n \geq 0$, we have

$$\|u_{ih} - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} \leq k^{n+1}\|u_h - u_{ih}^0\|_{L^\infty(\Omega)},\ i = 1, 2.$$

Proof. Using the proposition (3.1), where f is a Lipshizian function with rate α we have: for $i = 1, 2$

$$\|u_{ih} - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} = \|\sigma_h(f(u_{ih}) - \sigma_{ih}(f(u_h^n))\|_{L^\infty(\Omega_i)}$$

$$\leq \frac{1}{\beta}\|f(u_{ih}) - f(u_{ih}^n)\|_{L^\infty(\Omega_i)}$$

$$\leq \frac{\alpha}{\beta}\|u_{ih} - u_{ih}^n\|_{L^\infty(\Omega_i)}$$

and

$$\|u_{ih} - u_{ih}^n\|_{L^\infty(\Omega_i)} \leq \frac{1}{\beta}\|f(u_{ih}) - f(u_{ih}^{n-1})\|_{L^\infty(\Omega_i)}$$

$$\leq \frac{\alpha}{\beta}\|u_{ih} - u_{ih}^{n-1}\|_{L^\infty(\Omega_i)}$$

so

$$\|u_{ih} - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} \leq (\frac{\alpha}{\beta})^2\|u_{ih} - u_{ih}^{n-1}\|_{L^\infty(\Omega_i)}$$

$$\leq \cdots \leq \cdots \leq \left(\frac{\alpha}{\beta}\right)^{n+1}\|u_{ih} - u_{ih}^0\|_{L^\infty(\Omega_i)}.$$

Let $K = \dfrac{\alpha}{\beta}$ and by (2.5) we have $k \in (0, 1)$, then

$$\|u_{ih} - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} \leq k^{n+1}\|u_h - u_{ih}^0\|_{L^\infty(\Omega_i)}.$$

■

4. Approximation in L^∞-norm

In this section,we will improve the results which obtained in the previous works, further understanding to our previous work for the problem. We will use the functional approach, which is based on the lemma (3.1) which concerning the geometrical convergence, and an adaptation for the results concern the variational inequalities is due to Cortey-Dumont [9].

We recall that our discrete problem is the following:

$$\begin{cases} a(u_h, v - u_h) \geq (f(u_h), v - u_h) & \forall u_h, v \in K_{gh}(u_h) \\ u_h \leq r_h M u_h & \\ u_h = \pi_h g & \text{on } \partial\Omega \end{cases}$$

Putting $\overline{u_h}$ solution of the following equation:

$$a_h(\overline{u_h}, v_h) = a(u, v_h) \tag{4.1}$$

where u is the solution of the continuous problem (2.6). Due to Cortey-Dumont [9] we have the following results:

$$\begin{cases} \|u - \overline{u_h}\|_{L^\infty(\Omega)} \leq ch^2 |\log h|^2, \\ \|u - r_h u\|_{L^\infty(\Omega)} \leq ch^2 |\log h|^2, \end{cases} \tag{4.2}$$

by (2.4) we have:

$$\rho(x) = Mu(x)/_{B(x_0, ch)}, \tag{4.3}$$

where $B(x_0, ch)$ is the boulle of the center x_0 and the radius ch, ρ is the restriction of Mu on $B(x_0, ch)$ thus $\forall x \in B(x_0, ch)$ such that

$$u(x_0) = Mu(x_0) = \rho(x_0)$$

then

$$|u(x) - \rho(x)| \leq ch^2 || \log h|^2. \tag{4.4}$$

Remark 4.1. The nature of our problem is the existence of the free boundary between:

$$\Omega_0 = \{x \in \Omega / u(x) = Mu(x)\}$$

Ω_0 is the coincidence set. Let the following set, that is the discrete approximation of the coincidence set

$$\Omega_{0h} = \{x \in \cup T_h / T_h \cap \Omega_0 \neq \emptyset\},$$

where T_h is the element of the triangulation τ_h.

The Lemma that follows, is similar to the lemma given in [9] for the variational inequalities.

Lemma 4.2. Under the conditions in (2.1) to (2.5) and (4.2) to (4.4), we have the following estimates:

$$\|u - Mu\|_{L^\infty(\Omega)} \leq ch^2 |\log h|^2$$

$$\|Mu - r_h M u_h\|_{L^\infty(\Omega)} \leq ch^2 |\log h|^2.$$

Proof. The proof is an adaptation to [9] given T_h in Ω_h, there exists x_0 belonging to T_h such that $u(x_0) = Mu(x_0)$. Moreover $T_h \subset B(x_0, ch)$. So for every x in T_h $u(x) \leq$

$Mu(x)$. Since u is the solution of the quasi-variational inequality and from (4.3) and (4.4) we have:

$$u(x) \leq \rho(x) \leq u(x) + ch^2|\log h|^2 \leq Mu(x) + ch^2|\log h|^2,$$

on the other hand and by using (4.4) we obtain

$$\rho(x) \leq u(x) + ch^2|\log h|^2$$

and from (4.3) we get

$$\forall x \in B(x_0, ch) : Mu(x) \leq u(x) + ch^2|\log h|^2,$$

then

$$\|u - Mu\|_{L^\infty(\Omega)} \leq ch^2|\log h|^2,$$

the second estimate follows from (4.2) we have:

$$\|u - r_h u\|_{L^\infty(\Omega)} \leq ch^2|\log h|^2$$

and

$$r_h u \leq r_h Mu_h \leq -r_h u + ch^2|\log h|^2$$

so

$$\|Mu - r_h Mu_h\|_{L^\infty(\Omega)} = \|Mu - r_h Mu_h + u - u + r_h u - r_h u\|$$
$$\|Mu - u\| + \|u - r_h u\| + \|r_h u - r_h Mu_h\|$$

then

$$\|Mu - r_h Mu_h\| \leq ch^2|\log h|^2.$$

■

Using the previous results, we will present the lemma which plays an important roll in the main result.

Lemma 4.3. Under the conditions in (2.1) to (2.5) and (4.1) to (4.4), with discrete maximum principle, there exists a constant C_2 independents of h such that:

$$\|u_h - r_h Mu_h\|_{L^\infty(\Omega)} \leq C_2 h^2|\log h|^2.$$

Proof. Using the Theorem (2.1) and the Lemma (4.1) we find

$$\begin{aligned}\|u_h - r_h Mu_h\|_{L^\infty(\Omega)} &\leq \|u_h + u - u - Mu + Mu - r_h Mu_h\| \\ &\leq \|u_h - u\| + \|u - Mu\| + \|Mu - r_h Mu_h\| \\ &\leq C_2 h^2|\log h|^2.\end{aligned}$$

■

4.1. L^∞ error estimate

We finish by the main result.

Theorem 4.4. Under the same previous conditions there exists a constant C independents of h such that

$$\|u_i - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} \leq Ch^2|\log h|^2, \quad i = 1, 2.$$

Proof. For $i = 1, 2$ we have

$$\|u_i - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} \leq \|u_i - u_{ih}\| + \|u_{ih} - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)}$$

by using the Theorem 3.1 and the Lemma 3.2 we find

$$\|u_i - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} \leq C_1h^2|\log h|^2 + K^{n+1}\|u_h - u_{ih}^0\|_{L^\infty(\Omega_i)}$$

From to our choice $u_{ih}^0 = r_hMu_h$, and the Lemma 4.2 we get

$$\|u_i - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} \leq C_1h^2|\log h|^2 + K^{n+1}C_2h^2|\log h|^2$$
$$\leq (C_1 + K^{n+1}C_2)h^2|\log h|^2$$

then

$$\|u_i - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} \leq Ch^2|\log h|^2.$$

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