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Solution of the fourth order Cauchy Difference Equation on free groups

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Abstract

Let $f : G \to H$ be a function, where (G,.) is a group and (H,+) is an abelian group. In this paper, the following Fourth Order Cauchy difference of

$$\begin{aligned} f : C^{(4)} f(x_1, x_2, x_3, x_4, x_5) = {} & f(C_5(\prod_{i=1}^{5} x_i)) - f(C_4(\prod_{i=1}^{5} x_i)) \\ & + f(C_3(\prod_{i=1}^{5} x_i)) - f(C_2(\prod_{i=1}^{5} x_i)) \\ & + f(C_1(\prod_{i=1}^{5} x_i)) \forall x_1, x_2, x_3, x_4, x_5 \in G \end{aligned}$$

is studied. Where $f(C_r(\prod_{i=1}^{n} x_i))$ is defined as function of combination r at a time from n objects. We give some special solutions of $C^{(4)} f = 0$ on free groups.

AMS subject classification:
Keywords: Cauchy difference equation, free groups.

1. Introduction

It is well known from [1] that Jenson's functional equation

$$f(x + y) + f(x - y) = 2f(x) \tag{1.1}$$

with additional condition $f(0) = 0$, is equivalent to Cauchy's equation

$$f(x + y) = f(x) + f(y)$$

on the real line. Let (G, .) be a group, (H, +) is an abelian group. Let e $\in$ G and 0 $\in$ H denote identity elements. The study of (1.1) was extended groups for f maps G into H in [2], where the general solution for a free group H with two generators and $G = GL_n(z), n \geq 3$ (see[3]). Since the functional equations involve Cauchy difference, which made it become much more interesting [4–7]. For a function f: G→H, its cauchy difference $C^{(m)}f$, is defined by

$$C^{(0)}f = f, \tag{1.2}$$
$$C^{(1)}f(x_1, x_2) = f(x_1x_2) - f(x_1) - f(x_2) \tag{1.3}$$

$$C^{(m+1)}f(x_1, x_2, \ldots, x_{m+2}) = C^{(m)}f(x_1, x_2, x_3 \ldots, x_{m+2}) - C^{(m)}f(x_1, x_3, \ldots, x_{m+2}) - C^{(m)}f(x_2, x_3, \ldots, x_{m+2}) \tag{1.4}$$

The first order cauchy difference $C^{(1)}f$ will be abbreviated as Cf. In [9],by using the reduction formulas and relations, as given in [2,3], the general solution of third order Cauchy difference equation was provided on free groups.

In this paper, we consider the following functional equation:

$$f(C_5(\prod_{i=1}^{5} x_i)) - f(C_4(\prod_{i=1}^{5} x_i)) + f(C_3(\prod_{i=1}^{5} x_i)) - f(C_2(\prod_{i=1}^{5} x_i)) + f(C_1(\prod_{i=1}^{5} x_i)) = 0 \qquad \forall x_1, x_2, x_3, x_4, x_5 \in G \tag{1.5}$$

It follows from (1.4) that (1.5) is equivalent to the vanishing fourth order cauchy difference equation

$$C^{(4)}f = 0$$

The purpose of this paper is to determine the solutions of equation (1.5). The solution of equation (1.5) will be denoted by

$$KerC^{(4)}(G, H) = \{f : G \to H | f satisfies (1.5)\} \tag{1.6}$$

Remark 1.1.

1. $KerC^{(4)}(G, H)$ is an abelian group under the pointwise addition of functions;
2. $Hom(G, H) \subseteq KerC^{(4)}(G, H)$

2. Properties of Solutions

Lemma 2.1. Suppose that $f \in KerC^{(4)}(G, H)$. Then

$$\begin{aligned}
f(e) &= 0, && (2.1)\\
Cf(x, y) &= 0, \quad \text{when } x = e \text{ or } y = e && (2.2)\\
C^{(2)}f(x, y, z) &= 0, \quad \text{when } x = e \text{ or } y = e \text{ or } z = e && (2.3)\\
C^{(3)}f(x, y, z, u) &= 0, \quad \text{when } x = e \text{ or } y = e \text{ or } z = e \text{ or } u = e && (2.4)\\
C^{(3)}f & \quad \text{is a homomorphism with respect to each variable} && (2.5)\\
f(x^n) &= nf(x) + nC_2Cf(x, x) + nC_3C^{(2)}f(x, x, x)\\
&\quad + nC_4C^{(3)}f(x, x, x, x) && (2.6)
\end{aligned}$$

for all $x, y, z, u \in G$ and $n \in Z$.

Proof. Putting $x_1 = e$ in (1.5) we get (2.1).

$$\begin{aligned}
& f(x_2x_3x_4x_5) - f(x_2x_3x_4) - f(x_2x_3x_5) - f(x_2x_4x_5) - f(x_3x_4x_5)\\
&\quad - f(x_2x_3x_4x_5) + f(x_2x_3) + f(x_2x_4) + f(x_2x_5) + f(x_3x_4) + f(x_3x_5)\\
&\quad + f(x_4x_5) + f(x_2x_3x_4) + f(x_2x_3x_5) + f(x_2x_4x_5) + f(x_3x_4x_5)\\
&\quad - f(x_2) - f(x_3) - f(x_4) - f(x_5) - f(x_2x_3) - f(x_2x_4) - f(x_2x_5)\\
&\quad - f(x_3x_4) - f(x_3x_5) - f(x_4x_5) + f(e) + f(x_2) + f(x_3)\\
&\quad + f(x_4) + f(x_5) = 0
\end{aligned}$$

therefore f(e)=0.

Then from (2.1) we obtain (2.2)-(2.4)

$$\begin{aligned}
Cf(x, e) &= f(xe) - f(x) - f(e)\\
&= f(x) - f(x)\\
&= 0
\end{aligned}$$

Similarly we can obtain

$$\begin{aligned}
Cf(e, y) &= 0,\\
C^{(2)}f(e, y, z) &= f(eyz) - f(ey) - f(ez) - f(yz) + f(e) + f(y) + f(z)\\
&= 0,
\end{aligned}$$

Similarly we can obtain

$$
\begin{aligned}
C^{(2)} f(x, e, z) &= 0, \\
C^{(2)} f(x, y, e) &= 0, \\
C^{(3)} f(e, y, z, u) &= 0, \\
C^{(3)} f(x, e, z, u) &= 0, \\
C^{(3)} f(x, y, e, u) &= 0, \\
C^{(3)} f(x, y, z, e) &= 0.
\end{aligned}
$$

Furthermore, by the definition of $C^{(3)} f$, we have

$$
\begin{aligned}
C^{(3)} f(x, yw, z, u) \;\; = \;\; & f(xywzu) - f(xywz) - f(xywu) - f(xzu) - f(ywzu) \\
& + f(xyw) + f(xz) + f(xu) + f(ywz) + f(ywu) + f(zu) \\
& - f(x) - f(yw) - f(z) - f(u)
\end{aligned}
$$

and

$$
\begin{aligned}
& C^{(3)} f(x, y, z, u) + C^{(3)} f(x, w, z, u) \\
& \quad = f(xyzw) - f(xyz) - f(xzu) - f(xyu) - f(yzu) + f(xy) + f(xz) \\
& \quad + f(xu) + f(yz) + f(yu) + f(zu) - f(x) - f(y) - f(z) - f(u) \\
& \quad + f(xwzu) - f(xwz) - f(xwu) - f(xzu) - f(wzu) + f(xw) + f(xz) \\
& \quad + f(xu) + f(wz) + f(wu) + f(zu) - f(x) - f(w) - f(z) - f(u)
\end{aligned}
$$

One can easily check that

$$
C^{(3)} f(x, yw, z, u) - C^{(3)} f(x, y, z, u) - C^{(3)} f(x, w, z, u) = C^{(4)} f(x, y, w, z, u) = 0
$$

$$
\begin{aligned}
& C^{(3)} f(x, yw, z, u) - C^{(3)} f(x, y, z, u) - C^{(3)} f(x, w, z, u) \\
& \quad = f(xywzu) - f(xywz) - f(xywu) - f(xzu) \\
& \qquad - f(ywzu) + f(xyw) + f(xz) + f(xu) + f(ywz) \\
& \qquad + f(ywu) + f(zu) - f(x) - f(yw) - f(z) - f(u) \\
& \qquad - f(xyzw) + f(xyz) + f(xzu) + f(xyu) + f(yzu) \\
& \qquad - f(xy) - f(xz) - f(xu) - f(yz) - f(yu) - f(zu) \\
& \qquad + f(x) + f(y) + f(z) + f(u) - f(xwzu) + f(xwz)
\end{aligned}
$$

$$
\begin{aligned}
&+f(xwu)+f(xzu)+f(wzu)-f(xw)-f(xz)-f(xu)\\
&-f(wz)-f(wu)-f(zu)+f(x)+f(w)+f(z)+f(u)\\
=\ &f(xywzu)-f(xywz)-f(xywu)-f(ywzu)-f(xyzu)\\
&-f(xwzu)+f(xyw)+f(ywz)+f(ywu)+f(xyz)+f(xyu)\\
&+f(yzu)+f(xwz)+f(xwu)+f(xzu)+f(wzu)-f(yw)\\
&-f(xy)-f(yz)-f(yu)-f(xw)-f(xz)-f(xu)-f(wz)\\
&-f(wu)-f(zu)+f(y)+f(x)+f(w)+f(z)+f(u)\\
=\ &C^{(4)}f(x,y,w,z,u)\\
=\ &0 \quad by (1.5)
\end{aligned}
$$

Hence, the above relations imply the $C^{(3)}f(x,.,z,u)$ is a homomorphism. Similarly, the fact is also true for $C^{(3)}f(.,y,z,u)$, $C^{(3)}f(x,y,.,u)$ and $C^{(3)}f(x,y,z,.)$. This proves (2.5). ■

We now consider (2.6). Actually, it is trivial for n = 0,1,2,3 by (2.1) and by the definition of Cf. Suppose that (2.6) holds for all natural numbers smaller than n≥5, then

$$
\begin{aligned}
f(x^n) &= f(x^{n-3}xxx)\\
&= f(x^{n-3}xx)+f(x^{n-3}xx)+f(x^{n-3}xx)+f(xxx)-f(x^{n-3}x)\\
&\quad -f(x^{n-3}x)-f(x^{n-3}x)-f(xx)-f(xx)-f(xx)+f(x^{n-3})\\
&\quad +f(x)+f(x)+f(x)+C^{(3)}f(x^{n-3},x,x,x)\\
&= f(x^{n-1})+f(x^{n-1})+f(x^{n-1})+f(x^3)-f(x^{n-2})-f(x^{n-2})\\
&\quad -f(x^{n-2})-f(x^2)-f(x^2)+f(x^{n-3})+f(x)+f(x)+f(x)\\
&\quad +C^{(3)}f(x^{n-3},x,x,x)\\
&= 3f(x^{n-1})+f(x^3)-3f(x^{n-2})-3f(x^2)+f(x^{n-3})+3f(x)\\
&\quad +C^{(3)}f(x^{n-3},x,x,x)\\
&= 3\Big[(n-1)f(x)+(n-1)C_2Cf(x,x)+(n-1)C_3C^{(2)}f(x,x,x)\\
&\quad +(n-1)C_4C^{(3)}f(x,x,x,x)\Big]\\
&\quad +\big[3f(x)+3C_2Cf(x,x)+3C_3C^2f(x,x,x)\big]\\
&\quad -3\Big[(n-2)f(x)+(n-2)C_2Cf(x,x)+(n-2)C_3C^{(2)}f(x,x,x)\\
&\quad +(n-2)C_4C^{(3)}f(x,x,x,x)\Big]\\
&\quad -3\left[2f(x)+2C_2Cf(x,x)\right]
\end{aligned}
$$

$$+\Big[(n-3)f(x)+(n-3)C_2Cf(x,x)+(n-3)C_3C^{(2)}f(x,x,x)$$
$$+(n-3)C_4C^{(3)}f(x,x,x,x)\Big]$$
$$+\Big[3f(x)+(n-3)C^{(3)}f(x,x,x,x)\Big]$$
$$=nf(x)+nC_2Cf(x,x)+nC_3C^{(2)}f(x,x,x)+nC_4C^{(3)}f(x,x,x,x)$$

where the definition of $C^{(3)}f$ and (2.5) are used in the second equation. This gives (2.6) fora all $n \geq 0$. On the other hand, for any fixed integer $n > 0$, by (1.4) and (2.1), we have

$$\begin{aligned}
&C^{(3)}f(x^n,x^{-n},x^n,x^n)\\
&\quad= f(x^{2n})-f(x^n)-f(x^n)-f(x^{3n})-f(x^n)+f(e)\\
&\qquad+f(x^{2n})+f(x^{2n})+f(e)+f(e)+f(x^{2n})-f(x^n)\\
&\qquad-f(x^{-n})-f(x^n)-f(x^n)\\
&\quad=4f(x^{2n})-6f(x^n)-f(x^{3n})-f(x^{-n})\\
&\Rightarrow f(x^{-n})\\
&\quad=4f(x^{2n})-6f(x^n)-f(x^{3n})-C^{(3)}f(x^n,x^{-n},x^n,x^n)\\
&\quad=4\Big[2nf(x)+2nC_2Cf(x,x)+2nC_3C^{(2)}f(x,x,x)+2nC_4C^{(3)}f(x,x,x,x)\Big]\\
&\qquad-6\Big[nf(x)+nC_2Cf(x,x)+nC_3C^{(2)}f(x,x,x)+nC_4C^{(3)}f(x,x,x,x)\Big]\\
&\qquad-\Big[3nf(x)+3nC_2Cf(x,x)+3nC_3C^{(2)}f(x,x,x)+3nC_4C^{(3)}f(x,x,x,x)\Big]\\
&\qquad+n^4C^{(3)}f(x,x,x,x)\\
&\quad=\Big[-nf(x)+-nC_2Cf(x,x)+-nC_3C^{(2)}f(x,x,x)+-nC_4C^{(3)}f(x,x,x,x)\Big]
\end{aligned}$$

from (2.5) and the above claim for $n > 0$. This confirms (2.6) for $n < 0$. ■

Remark 2.2. For any function $f : G \to H$, the following statements are pairwise equivalent:

(i) The function $f \in KerC^{(4)}(G,H)$;

(ii) $C^{(3)}f(.,y,z,u)$ is a homomorphism;

(iii) $C^{(3)}f(x,.,z,u)$ is a homomorphism;

(iv) $C^{(3)}f(x,y,.,u)$ is a homomorphism;

(v) $C^{(3)}f(x,y,z,.)$ is a homomorphism;

Before presenting Proposition 2.4, we first introduce the following useful lemma, which was given in [8].

Lemma 2.3. (Lemma 2.4 in [8]) The following identity is valid for any function f:G→H and $l \in N$;

$$f(x_1x_2\dots x_l) = \sum_{m\le l}\ \sum_{1\le i_1<i_2<\dots<i_m\le l} C^{(m-1)} f(x_{i_1}, x_{i_2}, \dots, x_{i_m}) \tag{2.7}$$

Proposition 2.4. Suppose that $f \in KerC^{(4)}(G, H)$. Then

$$\begin{aligned} &f(x_1^{n_1}x_2^{n_2}\dots x_l^{n_l}) \\ &= \sum_{1\le i\le l}\Big[n_i f(x_i) + n_iC_2Cf(x_i, x_i) + n_iC_3C^{(2)} f(x_i, x_i, x_i) \\ &\quad + n_iC_4C^{(3)} f(x_i, x_i, x_i, x_i)\Big] \\ &\quad + \sum_{1\le i_1<i_2\le l} Cf(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}) + \sum_{1\le i_1<i_2<i_3\le l} n_{i_1}n_{i_2}n_{i_3}C^{(2)} f(x_{i_1}, x_{i_2}, x_{i_3}) \\ &\quad + \sum_{1\le i_1<i_2<i_3<i_4\le l} n_{i_1}n_{i_2}n_{i_3}n_{i_4}C^{(3)} f(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) \end{aligned} \tag{2.8}$$

for $n_i \in Z$ and all $x_i \in G$, $i = 1, 2, \dots l$ such that $x_j \ne x_{j+1}$, $j = 1, 2, \dots, l-1$.

Proof. Replacing x_i in (2.7) by $x_i^{n_i}$, we have

$$f(x_1^{n_1}x_2^{n_2}\dots x_l^{n_l}) = \sum_{m\le l}\ \sum_{1\le i_1<i_2<\dots<i_m\le l} C^{(m-1)} f(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}, \dots, x_{i_m}^{n_{i_m}})$$

The vanishing of $C^{(m-1)}f$ for $m \ge 5$ yields

$$\begin{aligned} f(x_1^{n_1}x_2^{n_2}\dots x_l^{n_l}) &= \sum_{1\le i\le l} f(x_i^{n_i}) + \sum_{1\le i_1<i_2\le l} Cf(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}) \\ &\quad + \sum_{1\le i_1<i_2<i_3\le l} C^{(2)} f(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}, x_{i_3}^{n_{i_3}}) \\ &\quad + \sum_{1\le i_1<i_2<i_3<i_4\le l} C^{(3)} f(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}, x_{i_3}^{n_{i_3}}, x_{i_4}^{n_{i_4}}) \end{aligned}$$

Therefore, by (2.6) and (2.5), we have

$$\begin{aligned} f(x_i^{n_i}) &= n_i f(x_i) + n_iC_2Cf(x_i, x_i) \\ &\quad + n_iC_3C^{(2)} f(x_i, x_i, x_i) + n_iC_4C^{(3)} f(x_i, x_i, x_i, x_i) \end{aligned}$$

$$\begin{aligned} C^{(2)}f(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}, x_{i_3}^{n_{i_3}}) &= n_{i_1}n_{i_2}n_{i_3}C^{(2)}f(x_{i_1}, x_{i_2}, x_{i_3}) \\ C^{(3)}f(x_{i_1}^{n_{i_1}}, x_{i_2}^{n_{i_2}}, x_{i_3}^{n_{i_3}}, x_{i_4}^{n_{i_4}}) &= n_{i_1}n_{i_2}n_{i_3}n_{i_4}C^{(3)}f(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) \end{aligned}$$

which is (2.8). This completes proof. ■

Remark 2.5. In particular, if $l = 1$, then Proposition 2.4 holds.

3. Solution on a free group

In this section, we study the solutions on free group. We first solve (1.5) for the free group G on a single letter a.

Theorem 3.1. Let G be the free group on one letter a. Then $f \in KerC^{(4)}(G, H)$ iff it is given by

$$\begin{aligned} f(a^n) &= nf(a) + nC_2Cf(a,a) + nC_3C^{(2)}f(a,a,a) \\ &\quad +nC_4C^{(3)}f(a,a,a,a) \quad \forall n \in Z \end{aligned} \tag{3.1}$$

Proof. Necessity. It can be obtained from (2.6) in Lemma 2.1.

Sufficiency. Taking (3.1) as the definition of f on G=$< a >$. By Remark 2.5, we only need to verify that $C^{(3)}f$ is a homomorphism with respect to each variable and thus f belongs to $KerC^{(4)}(G, H)$. Let

$$x = a^m, y = a^n, z = a^p, u = a^q$$

be any four elements of G. Then it follows from (1.4) and (3.1) that

$$\begin{aligned} &C^{(3)}f(x,y,z,u) = C^{(3)}f(a^m, a^n, a^p, a^q) \\ &= f(a^{m+n+p+q}) - f(a^{m+n+p}) - f(a^{m+n+q}) - f(a^{m+p+q}) \\ &\quad -f(a^{n+p+q}) + f(a^{m+n}) + f(a^{m+p}) + f(a^{m+q}) \\ &\quad +f(a^{n+p}) + f(a^{n+q}) + f(a^{p+q}) - f(a^m) - f(a^n) - f(a^p) - f(a^q) \\ &= N_1f(a) + N_1C_2Cf(a,a) + N_1C_3C^{(2)}f(a,a,a) + N_1C_4C^{(3)}f(a,a,a,a) \\ &\quad -N_2f(a) - N_2C_2Cf(a,a) - N_2C_3C^{(2)}f(a,a,a) - N_2C_4C^{(3)}f(a,a,a,a) \\ &\quad -N_3f(a) - N_3C_2Cf(a,a) - N_3C_3C^{(2)}f(a,a,a) - N_3C_4C^{(3)}f(a,a,a,a) \\ &\quad -N_4f(a) - N_4C_2Cf(a,a) - N_4C_3C^{(2)}f(a,a,a) - N_4C_4C^{(3)}f(a,a,a,a) \\ &\quad -N_5f(a) - N_5C_2Cf(a,a) - N_5C_3C^{(2)}f(a,a,a) - N_5C_4C^{(3)}f(a,a,a,a) \\ &\quad +N_6f(a) + N_6C_2Cf(a,a) + N_6C_3C^{(2)}f(a,a,a) + N_6C_4C^{(3)}f(a,a,a,a) \\ &\quad +N_7f(a) + N_7C_2Cf(a,a) + N_7C_3C^{(2)}f(a,a,a) + N_7C_4C^{(3)}f(a,a,a,a) \\ &\quad +N_8f(a) + N_8C_2Cf(a,a) + N_8C_3C^{(2)}f(a,a,a) + N_8C_4C^{(3)}f(a,a,a,a) \\ &\quad +N_9f(a) + N_9C_2Cf(a,a) + N_9C_3C^{(2)}f(a,a,a) + N_9C_4C^{(3)}f(a,a,a,a) \end{aligned}$$

$$+N_{10}f(a)+N_{10}C_2Cf(a,a)+N_{10}C_3C^{(2)}f(a,a,a)+N_{10}C_4C^{(3)}f(a,a,a,a)$$
$$+N_{11}f(a)+N_{11}C_2Cf(a,a)+N_{11}C_3C^{(2)}f(a,a,a)+N_{11}C_4C^{(3)}f(a,a,a,a)$$
$$-mf(a)-mC_2Cf(a,a)-mC_3C^{(2)}f(a,a,a)-mC_4C^{(3)}f(a,a,a,a)$$
$$-nf(a)-nC_2Cf(a,a)-nC_3C^{(2)}f(a,a,a)-nC_4C^{(3)}f(a,a,a,a)$$
$$-pf(a)-pC_2Cf(a,a)-pC_3C^{(2)}f(a,a,a)-pC_4C^{(3)}f(a,a,a,a)$$
$$-qf(a)-qC_2Cf(a,a)-qC_3C^{(2)}f(a,a,a)-qC_4C^{(3)}f(a,a,a,a)$$

Where $N_1=m+n+p+q$, $N_2=m+n+p$, $N_3=m+n+q$, $N_4=m+p+q$, $N_5=n+p+q$, $N_6=m+n$, $N_7=m+p$, $N_8=m+q$, $N_9=n+p$, $N_{10}=n+q$, $N_{11}=p+q$. By a tedious calculation, we have

$$C^{(3)}f(a^m,a^n,a^p,a^q)=mnpqC^{(3)}f(a,a,a,a)$$

which leads to the result that $C^{(3)}f$ is a homomorphism with respect to each variable. ■

At the end of this section, for the free group on an alphabet $\langle\mathscr{A}\rangle$ with $|\mathscr{A}|\geq 2$, we discuss some special solution of (1.5).

An element $x\in\mathscr{A}$ can be written in the form

$$x=a_1^{n_1}a_2^{n_2}\dots a_l^{n_l},\, where a_i\in\mathscr{A}, n_i\in Z \tag{3.2}$$

For each fixed $a\in\mathscr{A}$ and fixed pair of distinct $a,b\in\mathscr{A}$, define the functions W_1, W_2, W_3:

$$W_1(x;a)=\sum_{a_i=a}n_i \tag{3.3}$$

$$W_2(x;a,b)=\sum_{i<j,a_i=a,a_j=b}n_in_j \tag{3.4}$$

$$W_3(x;a,b)=\sum_{i>j,a_i=a,a_j=b}n_in_j \tag{3.5}$$

along with (3.2). Referring to [2,3], the above functions are well defined. Furthermore, they satisfy the following relations:

$$W_1(xy;a)=W_1(x;a)+W_1(y;a) \tag{3.6}$$

$$W_2(x;a,b)=W_3(x;b,a) \tag{3.7}$$

Proposition 3.2. For any fixed $a\in\mathscr{A}$ and fixed pair of distinct a,b in $\mathscr{A}$, the following assertions hold:

(i) $W_1(.;a)$ belongs to $KerC^{(4)}(\mathscr{A},Z)$;

(ii) $W_2(.;a)$ belongs to $KerC^{(4)}(\mathscr{A}, Z)$;

(iii) $W_3(.;a)$ belongs to $KerC^{(4)}(\mathscr{A}, Z)$;

Proof. Claim (i) follows from the fact that $x \to W(x;a)$ is a morphism from $\langle \mathscr{A} \rangle$ to Z by (3.6).

Now we consider assertion(ii). Let x,y,z,w,u in the free group be written as

$$\begin{aligned} x &= a_1^{r_1} a_2^{r_2} \dots a_l^{r_l}, \\ y &= b_1^{s_1} b_2^{s_2} \dots b_p^{s_p}, \\ z &= c_1^{t_1} c_2^{t_2} \dots c_q^{t_q}, \\ w &= d_1^{m_1} d_2^{m_2} \dots d_v^{m_v}, \\ u &= g_1^{n_1} g_2^{n_2} \dots g_k^{n_k}, \end{aligned}$$

Let

$$\begin{aligned} R &= \sum_{i<j, a_i=a, a_j=b} r_i r_j \\ S &= \sum_{i<j, b_i=a, b_j=b} s_i s_j \\ T &= \sum_{i<j, c_i=a, c_j=b} t_i t_j \\ M &= \sum_{i<j, d_i=a, d_j=b} m_i m_j \\ N &= \sum_{i<j, g_i=a, g_j=b} n_i n_j \\ R_s &= \sum_{a_i=a, b_j=b} r_i s_j \\ R_t &= \sum_{a_i=a, c_j=b} r_i t_j \\ R_m &= \sum_{a_i=a, d_j=b} r_i m_j \end{aligned}$$

$$\begin{aligned}
R_n &= \sum_{a_i=a, g_j=b} r_i n_j \\
S_t &= \sum_{b_i=a, c_j=b} s_i t_j \\
S_m &= \sum_{b_i=a, d_j=b} s_i m_j \\
S_n &= \sum_{b_i=a, g_j=b} s_i n_j \\
T_m &= \sum_{c_i=a, d_j=b} t_i m_j \\
T_n &= \sum_{c_i=a, g_j=b} t_i n_j \\
M_n &= \sum_{d_i=a, g_j=b} m_i n_j
\end{aligned}$$

Then

$$\begin{aligned}
W_2(xyzwu; a, b) &= R + S + T + M + N + R_s + R_t + R_m + R_n \\
&\quad + S_t + S_m + S_n + T_m + T_n + M_n \\
W_2(xyzw; a, b) &= R + S + T + M + R_s + R_t + R_m + S_t + S_m + T_m \\
W_2(xyzu; a, b) &= R + S + T + N + R_s + R_t + R_n + S_t + S_n + T_n \\
W_2(xywu; a, b) &= R + S + M + N + R_s + R_m + R_n + S_m + S_n + M_n \\
W_2(xzwu; a, b) &= R + T + M + N + R_t + R_m + R_n + T_m + T_n + M_n \\
W_2(yzwu; a, b) &= S + T + M + N + S_t + S_m + S_n + T_m + T_n + M_n \\
W_2(xyz; a, b) &= R + S + T + R_s + R_t + S_t \\
W_2(xyw; a, b) &= R + S + M + R_s + R_m + S_m \\
W_2(xyu; a, b) &= R + S + N + R_s + R_n + S_n \\
W_2(xzw; a, b) &= R + T + M + R_t + R_m + T_m \\
W_2(xzu; a, b) &= R + T + N + R_t + R_n + T_n \\
W_2(xwu; a, b) &= R + M + N + R_m + R_n + M_n \\
W_2(yzw; a, b) &= S + T + M + S_t + S_m + T_m
\end{aligned}$$

$$\begin{aligned}
W_2(yzu;a,b) &= S+T+N+S_t+S_n+T_n\\
W_2(ywu;a,b) &= S+M+N+S_m+S_n+M_n\\
W_2(zwu;a,b) &= T+M+N+T_m+T_n+M_n\\
W_2(xy;a,b) &= R+S+R_s\\
W_2(xz;a,b) &= R+T+R_t\\
W_2(xw;a,b) &= R+M+R_m\\
W_2(xu;a,b) &= R+N+R_n\\
W_2(yz;a,b) &= S+T+S_t\\
W_2(yw;a,b) &= S+M+S_m\\
W_2(yu;a,b) &= S+N+S_n\\
W_2(zw;a,b) &= T+M+T_m\\
W_2(zu;a,b) &= T+N+T_n\\
W_2(wu;a,b) &= M+N+M_n\\
W_2(x;a,b) &= R\\
W_2(y;a,b) &= S\\
W_2(z;a,b) &= T\\
W_2(w;a,b) &= M\\
W_2(u;a,b) &= N
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&W_2(xyzwu;a,b) - W_2(xyzw;a,b) - W_2(xyzu;a,b) - W_2(xywu;a,b)\\
&\quad - W_2(xzwu;a,b) - W_2(yzwu;a,b) + W_2(xyz;a,b) + W_2(xyw;a,b)\\
&\quad + W_2(xyu;a,b) + W_2(xzw;a,b) + W_2(xzu;a,b) + W_2(xwu;a,b)\\
&\quad + W_2(yzw;a,b) + W_2(yzu;a,b) + W_2(ywu;a,b) + W_2(zwu;a,b)\\
&\quad - W_2(xy;a,b) - W_2(xz;a,b) - W_2(xw;a,b) - W_2(xu;a,b)\\
&\quad - W_2(yz;a,b) - W_2(yw;a,b) - W_2(yu;a,b) - W_2(zw;a,b)\\
&\quad - W_2(zu;a,b) - W_2(wu;a,b) + W_2(x;a,b) + W_2(y;a,b) + W_2(z;a,b)\\
&\quad + W_2(w;a,b) + W_2(u;a,b) = 0
\end{aligned}$$

This concludes assertion (ii). Claims (iii) follows from (3.7) directly. ■

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