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Product of Adjoint of a Weighted Composition Operator with a Weighted Composition Operator on the Fock Space

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Abstract

We study weighted composition operators on the Fock space of analytical functions on $\mathbb{C}$ with kernels of the form $e^{\frac{\langle z,w\rangle}{2}}$**.** In this paper we obtain several results of Product of Adjoint of a Weighted Composition Operator with a Weighted Composition Operator on the Fock Space of $\mathbb{C}$.

AMS Subject Classification: Primary 47B38

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1. Introduction

Throughout this paper, let dA(z) denote the usual Lebesgue measure in $\mathbb{C}$. The Fock space $\mathcal{F}^2$, is also known as the Segal-Bergmann space, consists of all analytical functions f on complex plane $\mathbb{C}$ that are square integrable with respect to Gaussian measure $d\mu = (2\pi)^{-1}e^{-\frac{|z|^2}{2}}dA(z)$ for which $\|f\| = \frac{1}{2\pi}\int_{\mathbb{C}} |f(z)|^2 e^{-\frac{|z|^2}{2}}\, dm(z)$

The space $\mathcal{F}^2$ is a reproducing kernel Hilbert space with inner product

$\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbb{C}} f(z)\overline{g(z)}e^{-\frac{|z|^2}{2}}\, dA(z),\ f, g \in \mathcal{F}^2$

The reproducing kernel function for $\mathcal{F}^2$ is given by $K_w(z) = e^{\frac{\langle z,w\rangle}{2}}$. We also use the normalized kernel function $k_w(z) = e^{\frac{\langle z,w\rangle}{2}-\frac{|w|^2}{4}}$.

Some problems related to Fock space have been studied by many authors [1, 7, 9].

Our object of investigation is weighted composition operator $W_{\psi,\phi}f = \psi.f \circ \phi = \psi C_\phi$, where ψ is analytical function on $\mathbb{C}$ and ϕ is analytical self map on $\mathbb{C}$. When $\psi(z) = 1$, C_ϕ is called composition operator.
The boundedness and compactness of C_ϕ on Fock space have been studied [2,6]. Also boundedness and compactness of ψC_ϕ have been studied by [8].
Books [3, 4, 5] are good reference for basic theory.
In section 2, we collect some important results, and notation which we will need in the proof of our results. In section 3, we characterize Product of Adjoint of a Weighted Composition Operator with a Weighted Composition Operator on the Fock Space.

2. Preliminaries

Lemma 2.1: The following result is well known and easy to verify.
Let $z \in \mathbb{C}$. Since
$C_\phi K_w(z) = K_w(\phi(z)) = \overline{K_{\phi(z)}(w)}$,
We have
$C_\phi^* K_z(w) = \langle K_z, C_\phi K_w \rangle = K_{\phi(z)}(w)$.

Lemma 2.2: The following results are well known and easy to verify.
$W_{\psi,\phi}^* K_w(z) = \overline{\psi(w)} K_{\phi(w)}(z)$
$W_{\psi,\phi} K_w(z) = \psi(z) K_w \phi(z)$

Theorem 2.3 [7]: Let $\phi: \mathbb{C}^n \to \mathbb{C}^n$ be analytical mapping. The following statements hold:

a) If C_ϕ is bounded on $\mathcal{F}^2$, then $\phi(z) = Az + B$, where A is an $n \times n$ matrix and B is an $n \times 1$ vector. Furthermore, $\|A\| \leq 1$, and if $|A\xi| = |\xi|$ for some ξ in C^n, then $\langle A\xi, B \rangle = 0$.
b) If C_ϕ is compact on $\mathcal{F}^2$, then $\phi(z) = Az + B$, where $\|A\| < 1$.

Note that when n=1, (a) simply says that if C_ϕ is bounded on $\mathcal{F}^2$, then $\phi(z) = az + b$, where $|a| \leq 1$, and if $|a| < 1$, then b=0.

Theorem 2.4 [10]: Let ψ and ϕ be entire functions on $\mathbb{C}$ such that f is not identically zero. Then $W_{\psi,\phi}$ is bounded on $\mathcal{F}^2$ if and only if ψ belong to $\mathcal{F}^2$, $\phi(z) = \phi(0) + \lambda z$ with $|\lambda| \leq 1$ and $M(\psi, \phi) \coloneqq \sup\{|\psi|^2 \exp(|\phi(z)|^2 - |z|^2; z \in \mathbb{C}\} < \infty$.

Proposition 2.5 [1]: For $q \in \mathbb{C}$, denote $\phi_q(z) = q - z$, $U_q = W_{k_q,\phi_q}$, where $k_q(z) = \exp\frac{\langle z,q\rangle}{2} - \frac{|q|^2}{4}$, normalized kernel function. The operator $U_q = W_{k_q,\phi_q}$ is an unitary operator on $\mathcal{F}^2$.

Theorem 2.6 [1]: Let $\phi(0) = 0$. $W_{\psi,\phi}$ is unitary operator on $\mathcal{F}^2$ if and only if there exist an unitary operator $A: \mathbb{C}^n \to \mathbb{C}^n$ and a constant $\alpha \in \mathbb{C}$, $|\alpha| = 1$ such that $\phi(z) = Az$ and $\psi(z) = \alpha$.

Note that when n=1, the above theorem can be written as $W_{\psi,\phi}$ is unitary operator on $\mathcal{F}^2$ if and only if there exists $\lambda, \alpha \in \mathbb{C}$, such that $\phi(z) = \lambda z$ and $\psi(z) = \alpha$ where $|\lambda| = |\alpha| = 1$.

Lemma 2.7 [11]: For $p \in \mathbb{C}$, denote $\phi_p(z) = p - z, U_p = W_{k_p,\phi_p}$, where $k_p(z) = \exp\frac{\langle z,p\rangle}{2} - \frac{|p|^2}{4}$, normalized kernel function. The operator $U_p = W_{k_p,\phi_p}$ is Hermitian on $\mathcal{F}^2$.
The following theorems are our main results.

3. Product of Adjoint of a Weighted Composition Operator with a Weighted Composition Operator

Lemma 3.1: C_ϕ is invertible on $\mathcal{F}^2$ if and only if ϕ is invertible.

Proof: Assume that C_ϕ is invertible.
Then their exist an analytic function ψ on $\mathbb{C}$ such that $C_\phi C_\psi = 1$.
This implies $C_\phi C_\psi K_w(z) = K_w(z)$
$K_w \phi \circ \psi(z) = K_w(z)$
$e^{\frac{\langle \phi\circ\psi(z),w\rangle}{2}} = e^{\frac{\langle z,w\rangle}{2}}$
$\langle \phi \circ \psi(z), w\rangle = \langle z, w\rangle$
This implies ϕ is invertible.
Conversely assume that ϕ is invertible with inverse ϕ^{-1} (say)
Consider
$C_\phi C_{\phi^{-1}} K_w(z) = K_w \phi \circ \phi^{-1}(z)$
$= e^{\frac{\langle \phi\circ\phi^{-1}(z),w\rangle}{2}}$
$= e^{\frac{\langle z,w\rangle}{2}}$
$= K_w(z)$
Hence C_ϕ is invertible and inverse of C_ϕ is $C_{\phi^{-1}}$.
That is $C_\phi^{-1} = C_{\phi^{-1}}$.
Now assume that ϕ, ψ are entire functions on $\mathbb{C}$. In the following theorem we discuss when the adjoint of a composition operator on $\mathcal{F}^2$ is inverse another composition operator. We discuss necessary and sufficient condition for which $C_\phi C_\psi^* = I$ on $\mathcal{F}^2$.

Proposition 3.2: Let C_ϕ, C_ψ be bounded on $\mathcal{F}^2$. Then $C_\phi C_\psi^*$ is invertible.

Proof: Since C_ϕ bounded on $\mathcal{F}^2$, by **Theorem 2.3 [7],** ϕ can be written in the form $\phi(z) = az + b$ for some $a, b \in \mathbb{C}$.
This implies ϕ is one-one and onto.
Hence ϕ is invertible.
By **Lemma 3.1,** C_ϕ is invertible and ${C_\phi}^{-1} = C_{\phi^{-1}}$.
Similarly, C_ψ is invertible and ${C_\psi}^{-1} = C_{\psi^{-1}}$.
Therefore ${C_\psi}^*$ is invertible.
Consider
$C_\phi {C_\psi}^* C_{\phi^{-1}} {C_{\psi^{-1}}}^* K_w(z) = K_{\psi\circ\psi^{-1}(w)} \phi \circ \phi^{-1}(z)$
$= K_w z$
This implies $C_\phi {C_\psi}^*$ is invertible.
Hence $C_\phi {C_\psi}^* = I$ on $\mathcal{F}^2$.

Corollary 3.3: Let ϕ, ψ are entire functions on $\mathbb{C}$. If $C_\phi {C_\psi}^*$ is invertible, then there exist analytic functions ζ, τ on $\mathbb{C}$ such that $\phi(\zeta(z)) = az$, $\psi(\tau(z)) = bz$ and $a\bar{b} = 1$ for some $a, b \in \mathbb{C}$.

Proof: Assume $C_\phi {C_\psi}^*$ is invertible then there exist $C_\zeta {C_\tau}^*$ such that
$C_\phi {C_\psi}^* C_\zeta {C_\tau}^* = I$
$C_\phi {C_\psi}^* C_\zeta {C_\tau}^* K_w(z) = K_w(z)$
$K_{\psi\circ\tau(w)} \phi \circ \zeta(z) = K_w(z)$
$e^{\frac{\langle \phi\circ\zeta(z), \psi\circ\tau(w)\rangle}{2}} = e^{\frac{\langle z,w\rangle}{2}}$
$\langle \phi \circ \zeta(z), \psi \circ \tau(w)\rangle = \langle z, w\rangle$
Since composition of two analytic functions is analytic, we have $\phi \circ \zeta(z) = \sum_{n=0}^{\infty} a_n z^n$, and $\psi \circ \tau(w) = \sum_{n=0}^{\infty} b_n w^n$ where $a_n, b_n \in \mathbb{C}$.
From the above equation, we get

$$\sum_{n=0}^{\infty} a_n z^n \overline{\sum_{n=0}^{\infty} b_n w^n} = z\bar{w}$$

Equating coefficient and taking $a_1 = a$ and $b_1 = b$, we conclude that $a\bar{b} = 1$.

Lemma 3.4: Let ϕ be entire functions on $\mathbb{C}$ and f be analytic functions such that ϕ fix the origin and f= α with $|\alpha| = 1, \alpha \in \mathbb{C}$. Then $W_{f,\phi}$ bounded on $\mathcal{F}^2$if and only if $W_{f,\phi}$ is unitary on $\mathcal{F}^2$.

Proof: Let $W_{f,\phi}$ bounded on $\mathcal{F}^2$.
By **Theorem 2.4**, $W_{f,\phi}$ is bounded on $\mathcal{F}^2$ if and only if f belong to $\mathcal{F}^2$, $\phi(z) = \phi(0) + \lambda z$ with $|\lambda| \leq 1$.
Since $\phi(0) = 0$, we get $\phi(z) = \lambda z$ with $|\lambda| \leq 1$.
By **Theorem 2.5**, $W_{f,\phi}$ is unitary on $\mathcal{F}^2$
The sufficiency is easy to verify.

Now assume that ϕ, ψ are entire functions on $\mathbb{C}$ and f, g are analytic functions such that $W_{f,\phi}$ and $W_{g,\psi}$ are bounded on $\mathcal{F}^2$. In the following theorem we discuss when the adjoint of a weighted composition operator on $\mathcal{F}^2$ is inverse another weighted composition operator. We discuss necessary and sufficient condition for which $W_{f,\phi}{W_{g,\psi}}^* = I$ on $\mathcal{F}^2$.

Theorem 3.5: Let $W_{f,\phi}$ and $W_{g,\psi}$ are bounded on $\mathcal{F}^2$ with $\psi(0) = \phi(0) = 0$. Then $W_{f,\phi}{W_{g,\psi}}^* = I$ if and only if there exists constant functions f, g such that $f\bar{g} = 1$ and $\lambda, \mu \in \mathbb{C}$ such that $\lambda\bar{\mu} = 1$.

Proof: Consider

$$W_{f,\phi}W_{g,\psi}^* = I$$

$$W_{f,\phi}W_{g,\psi}^* K_w(z) = K_w(z)$$

$$\overline{g(w)}f(z)K_{\psi(w)}\phi(z) = K_w(z) \qquad [1]$$

Put z=0, we get

$$\overline{g(w)}f(0)K_{\psi(w)}\phi(0) = K_w(0)$$

Since $\phi(0) = 0$, we have

$$\overline{g(w)}f(0) = 1$$

This implies

$$\overline{g(w)} = \frac{1}{f(0)} \qquad [2]$$

Thus g is a constant function on $\mathbb{C}$.

Put w=0 in [1]

$$\overline{g(0)}f(z)K_{\psi(0)}\phi(z) = K_0(z)$$

This implies

$$\overline{g(0)}f(z) = 1$$

$$f(z) = \frac{1}{\overline{g(0)}}$$

From [2],

$$f(z) = \frac{1}{\left(\frac{1}{f(0)}\right)} = f(0) \qquad [3]$$

Hence f is constant function on $\mathbb{C}$.

From [2] and [3], we get

$$f(z)\overline{g(w)} = 1$$

Therefore [1] implies

$$K_{\psi(w)}\phi(z) = K_w(z)$$

$$e^{\frac{\langle \phi(z),\psi(w)\rangle}{2}} = e^{\frac{\langle z,w\rangle}{2}}$$

Since $W_{f,\phi}$ and $W_{g,\psi}$ are bounded on $\mathcal{F}^2$ with $\psi(0) = \phi(0) = 0$, by Theorem 2.4,there exists $\lambda, \mu \in \mathbb{C}$ such that

$$e^{\frac{\langle \lambda z,\mu w\rangle}{2}} = e^{\frac{\langle z,w\rangle}{2}}$$

$$\langle \lambda z, \mu w\rangle = \langle z, w\rangle$$

$$\lambda\bar{\mu}\langle z, w\rangle = \langle z, w\rangle$$

$\lambda\bar{\mu} = 1$
Hence the proof.
The converse is easy to prove.
Now assume that ϕ, ψ are entire functions on $\mathbb{C}$ and f, g are analytic functions such that $W_{f,\phi}$ and $W_{g,\psi}$ are bounded on $\mathcal{F}^2$. In the following theorem we discuss when the adjoint of a weighted composition operator on $\mathcal{F}^2$ is another weighted composition operator. We determine the necessary and sufficient condition for which $W_{f,\phi} = W_{g,\psi}{}^*$ on $\mathcal{F}^2$.

Corollary 3.6: Let $W_{f,\phi}$ and $W_{g,\psi}$ are bounded on $\mathcal{F}^2$ with $\psi(0) = \phi(0) = 0$. Then $W_{f,\phi} = W^*_{g,\psi}$ on $\mathcal{F}^2$, if and only if there exists constant functions f, g such that $f = \bar{g}$ and $\lambda, \mu \in \mathbb{C}$ such that $\lambda = \bar{\mu}$.

Proof: Consider
$W_{f,\phi} = W^*_{g,\psi}$
$W_{f,\phi} K_w(z) = W^*_{g,\psi} K_w(z)$
[1] $f(z) e^{\frac{\langle \phi(z), w\rangle}{2}} = \overline{g(w)} e^{\frac{\langle z, \psi(w)\rangle}{2}}$
Take w=0 in (1), we get
[2] $f(z) = \overline{g(0)}$
Take z=0 in (1), we get
[3] $\overline{g(w)} = f(0)$
Thus, g is a constant function.
Substitute [2] and [3] in [1], we get
$\overline{g(0)} e^{\frac{\langle \phi(z), w\rangle}{2}} = f(0) e^{\frac{\langle z, \psi(w)\rangle}{2}}$
Since $f(0) = \overline{g(0)}$ from [3]
$e^{\frac{\langle \phi(z), w\rangle}{2}} = e^{\frac{\langle z, \psi(w)\rangle}{2}}$
$\langle \phi(z), w\rangle = \langle z, \psi(w)\rangle$
Since $W_{f,\phi}$ and $W_{g,\psi}$ are bounded on $\mathcal{F}^2$ with $\psi(0) = \phi(0) = 0$, by Theorem 2.4,there exists $\lambda, \mu \in \mathbb{C}$ such that
$\langle \lambda z, w\rangle = \langle z, \mu w\rangle$
$\lambda\langle z, w\rangle = \bar{\mu}\langle z, w\rangle$
Therefore $\lambda = \bar{\mu}$
Further, since $\psi(0) = 0$, we have from [1], [3]
$f(z) e^{\frac{\langle \phi(z), w\rangle}{2}} = \overline{g(w)} e^{\frac{\langle z, \psi(w)\rangle}{2}}$
$f(z) e^{\frac{\langle \phi(z), w\rangle}{2}} = f(0) e^{\frac{\langle z, \psi(w)\rangle}{2}}$
Taking w=0, we have
$f(z) = f(0)$
This implies f is a constant function.
From [2], we get
$\overline{g(w)} = f(z)$
(i.e) $f = \bar{g}$

Hence the proof.
The converse is easy to verify.

Corollary 3.7:
Let $W_{f,\phi}$ and $W_{g,\psi}$ are bounded on $\mathcal{F}^2$ with $\phi(0) = \psi(0) = p$ for some $p \in \mathbb{C}$. Let f, g be analytic functions on $\mathbb{C}$ and let ψ, ϕ be analytic self map on $\mathbb{C}$. Then $W_{f,\phi}W^*_{g,\psi} = I$ on $\mathcal{F}^2$, if and only if there is constant $\eta \neq 0$ such that $f = \frac{\eta}{k_p o\phi}$ and $g = \frac{\bar{\eta}}{(k_p o\psi)}$.

Proof: Consider
$W_{f,\phi}W^*_{g,\psi} = I$
Let us define $\phi_p = \psi_p = p - z, U_p = W_{k_p,\phi_p}$, $p \in \mathbb{C}$,
Consider
$W_{f,\phi}U_p(z) = W_{f,\phi}W_{k_p,\phi_p}(z)$
$= W_{f,\phi}k_p\phi_p(z)$
$= f.\, k_p o\phi.\, \phi_p o\phi(z)$
$= W_{\tilde{f},\tilde{\phi}}(z)$
Where $\tilde{f} = f.\, k_p o\phi$ and $\tilde{\phi} = \phi_p o\phi$
Similarly, we have
$W_{g,\psi}U_p(z) = W_{\tilde{g},\tilde{\psi}}(z)$
Where $\tilde{g} = g.\, k_p o\psi$ and $\tilde{\psi} = \psi_p o\psi$
By **Proposition 2.5 [7],** U_p is an unitary operator on $\mathcal{F}^2$, we get
$W_{f,\phi}U_pW_{g,\psi}{}^*U_p{}^* = I$
$W_{\tilde{f},\tilde{\phi}}W_{\tilde{g},\tilde{\psi}}{}^* = I$
Also $\tilde{\phi}(0) = \phi_p o\phi(0) = \phi_p(p) = 0$ and $\tilde{\psi}(0) = \psi_p o\psi(0) = \psi_p(p) = 0$
By **Theorem 3.5**, there exists constant functions $\tilde{f}$, $\tilde{g}$ such that $\tilde{f} = \bar{\tilde{g}}$.
Suppose $\tilde{f} = \eta \neq 0$ and $\tilde{g} = \bar{\eta}$ then $f = \frac{\tilde{f}}{k_p o\phi} = \frac{\eta}{k_p o\phi}$ and $g = \frac{\bar{\eta}}{(k_p o\psi)}$.
Converse is easy to verify. Hence the theorem.

Corollary 3.8: Let $W_{f,\phi}$ and $W_{g,\psi}$ are bounded on $\mathcal{F}^2$ with $\phi(0) = \psi(0) = p$ for some $p \in \mathbb{C}$. Let f, g be analytic functions on $\mathbb{C}$ and let ψ, ϕ be analytic self map on $\mathbb{C}$. Then $W_{f,\phi} = W^*_{g,\psi}$ on $\mathcal{F}^2(\mathbb{C}^n)$, if and only if there is constant $\eta \neq 0$ such that $f = \frac{\eta}{k_p o\phi}$ and $g = \frac{\bar{\eta}}{(k_p o\psi)}$.

Proof: Consider
$W_{f,\phi} = W^*_{g,\psi}$,
Let us define $\phi_p = \psi_p = p - z, U_p = W_{k_p,\phi_p}$, $p \in \mathbb{C}$,
Consider
$W_{f,\phi}U_p(z) = W_{f,\phi}W_{k_p,\phi_p}(z)$
$= W_{f,\phi}k_p\phi_p(z)$

$= f.k_p o\phi.\phi_p o\phi(z)$
$= W_{\tilde{f},\tilde{\phi}}(z)$
Where $\tilde{f} = f.k_p o\phi$ and $\tilde{\phi} = \phi_p o\phi$
Similarly, we have
$W_{g,\psi}U_p(z) = W_{\tilde{g},\tilde{\psi}}(z)$
Where $\tilde{g} = g.k_p o\psi$ and $\tilde{\psi} = \psi_p o\psi$
By **Lemma 2.7[11],** U_p is Hermitian operator on $\mathcal{F}^2$, we get
$W_{f,\phi}U_p = {W_{g,\psi}}^* {U_p}^*$
$W_{\tilde{f},\tilde{\phi}} = {W_{\tilde{g},\tilde{\psi}}}^*$
Also $\tilde{\phi}(0) = \phi_p o\phi(0) = \phi_p(p) = 0$ and $\tilde{\psi}(0) = \psi_p o\psi(0) = \psi_p(p) = 0$
By Corollary 3.3, there exists constant functions $\tilde{f}$, $\tilde{g}$ such that$\tilde{f} = \overline{\tilde{g}}$.
Suppose $\tilde{f} = \eta \neq 0$ and $\tilde{g} = \overline{\eta}$ then $f = \frac{\tilde{f}}{k_p o\phi} = \frac{\eta}{k_p o\phi}$ and $g = \frac{\overline{\eta}}{(k_p o\psi)}$.
Converse is easy to verify. Hence the theorem.

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