Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 6 (2015), pp. 4395-4403

http://www.ripublication.com

Stochastic Order Relationship and Its Applications in Actuarial Science

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Abstract

This paper focus on stochastic order and comparing risks. To this end, we deal some relationship of different stochastic order. Some examples and applications in actuarial science are given.

Keywords: Convex order, Comonotonicity, Lorenz order.

2000 MSC: Primary 62E17; Secondary 62P05.

Introduction

Stochastic ordering has been used successfully to solve various problems in applied probability, financial derivatives and risk measures. Once the validity of such a relation is established, it can be exploited to derive a host of inequalities among various quantities. The objective of actuarial theory and practice is comparing and ordering risks. In this work we use some relationship of different stochastic order. In general the random variables in an insurance portfolio are assumed to be mutually independently. To this end, we present some examples and applications in actuarial science are given. Despite the importance of this problem for practical applications, we will not discuss these issues in this paper.
The paper is organized as follows: In section 2 we introduce the preliminaries and the notations and we will recall some basic concepts and lemmas which will be used in later sections. Section 3 is devoted to state the main results and its proofs. Finally, we give some examples and application of the theory of ordering risks in modern actuarial science.

Preliminaries and notations

In this section we will collect some definitions and notations of different stochastic orders.

Definition 1(Convex order). Given two rvs X and Y such that $\mathbb{E}[X] = \mathbb{E}[Y]$, X is said to be smaller than Y in the convex order, written as $X \preccurlyeq_{CX} Y$, if π_X (t) $\leq \pi_Y$(t) for all $t \in \mathbb{R}$.

Definition 2 (Stop-loss transform). The function $\pi_X(t) = \mathbb{E}[(X - t)_+]$ is called the stop-loss transform of X (See Kaas 1993)

Definition 3 (Value-at- Risk). Given a risk X and a probability level $p \in (0,1)$, the corresponding VaR, denoted by $VaR[X;p]$, is defined as
$VaR[X;p] = F_X^{-1}(p)$
See Groovaerts et al. (1984).

Remark 1. *For all* $x \in \mathbb{R}$ *and* $p \in (0,1)$: $VaR[X;p] \leq x \Leftrightarrow p \leq F_X(x)$

Definition 4 (Tail value-At- Risk). Given a risk X and a probability level p, the corresponding $TVaR$, denoted by $TVaR[X;p]$, is defined as

$$TVaR[X;p] = \left(\frac{1}{(1-p)}\right)\int_p^1 VaR[X;\xi]d\xi, \quad 0 < p < 1$$

Definition 5 (Stochastic dominance). Let X and Y be too random variables. Then X is said to be smaller than Y in stochastic dominance, denoted as $X \leq_{ST} Y$, if the inequality $VaR[X;p] \leq VaR[Y;p]$ is satisfied for all $p \in [0,1]$. (See Lehmann (1959), Marshall and Olkin (1979) Ross (1983) and Stoyan (1983)).

Proposition 1. Given any rvs X and Y, the following equivalences hold:

1. $X \leq_{ST} Y \Leftrightarrow F_X(t) \geq F_Y(t)\ for\ all\ t \in \mathbb{R}$.
2. $X \leq_{ST} Y \Leftrightarrow \mathbb{E}[v(X)] \leq \mathbb{E}[v(Y)]$ for all non-decreasing function v, such that the expectations exist.
3. $X \leq_{ST} Y \Leftrightarrow \mathbb{E}[v(X)] \leq \mathbb{E}[v(Y)]$ for all functions v with $v' \geq 0$ such that the expectations exist.
4. $X \leq_{ST} Y \Leftrightarrow -Y \leq_{ST} -X$.

Proof. See [3]

Lorenz curves
The lorenz order is defined by means of pointwise comparison of the lorenz curves. Moreover, let X be a non-negative random variable df F_X. The lorenz curve LC_X is defined by

$$LC_X = \left(\frac{1}{\mathbb{E}(X)}\right)\int_{\xi=0}^{p} VaR[X;\xi]d\xi, \quad p \in [0,1].$$

such that the expectation exist.

Lorenz order

Definition 6. Consider two risks X and Y. Then, X is said to be smaller than Y with lorenz order, henceforth denoted by $X \preccurlyeq_{lorenz} Y$, when $LC_X(p) \geq LC_Y(p)$ for all $p \in [0,1]$.(See Arnold (1987))

Lorenz and convex orders

Proposition 2. Given two risks X and Y

$$X \preccurlyeq_{lorenz} Y \Leftrightarrow \left(\frac{X}{\mathbb{E}(X)}\right) \preccurlyeq_{CX} \left(\frac{Y}{\mathbb{E}(Y)}\right)$$

Proof. See [3]

Corollary 1. *Given two risks* X *and* Y, *if* $\mathbb{E}(X) = \mathbb{E}(Y)$, *we have from proposition (2) that*:

$X \preccurlyeq_{lorenz} Y \Leftrightarrow X \preccurlyeq_{CX} Y$

Remark 2. *We can see clearly that convex and lorenz orders are closely related.*

Majorization

Definition 7. Let $x, y \in (\mathbb{R}^{+})^{n}$. We say that y majorizes, and write $x \preccurlyeq_{MAJ} y$, if $\sum_{i=1}^{k} x_{(i;n)} \geq \sum_{i=1}^{k} y_{(i;n)}$ for $k = 1,2,\ldots,n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$.

It is easy to see that $x \preccurlyeq_{MAJ} y$ is equivalent to

$\sum_{i=1}^{k} x_{(i;n)} \leq \sum_{i=1}^{k} y_{(i;n)}$ for $k = 1,2,\ldots,n$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$.

Corollary 2. *We can see that* $y \preccurlyeq_{MAJ} y$, *with*

$$\bar{y} = (\bar{y}, \bar{y}, \ldots, \bar{y}) \text{ where } y = \left(\frac{1}{n}\right) \sum_{i=1}^{n} y_i,$$

Proposition 3. For all convex function $g: \mathbb{R} \to \mathbb{R}$ we have:

1. $\sum_{i=1}^{n} g(\bar{y}) \leq \sum_{i=1}^{n} g(y_i)$
2. $\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i)$

Proof. See [3]

Majorization and convex order

Corollary 3. *Let* $X_1, X_2, \ldots, X_n$ *be independent and identically distributed rvs. Let* a *and* b *be two vectors of constants. If* $a \preccurlyeq_{MAJ} b$ *then*

$$\sum_{i=1}^{n} a_i X_i \preccurlyeq_{CX} \sum_{i=1}^{n} b_i X_i.$$

Some properties of convex order

Lemma 1. *The convex order is closed under Convolution: Let* $X_1, X_2, \dots, X_m$ *be a set of independent random variable and* $Y_1, Y_2, \dots, Y_m$ *be another set of independent random variables. If* $X_i \le_{cx} Y_i$, *for* $i = 1, \dots, m$, *then* $\sum_{j=1}^{m} X_j \le_{cx} \sum_{j=1}^{m} Y_j$.

Proof. See [3]

Corollary 4. *Let* X *be a random variable with finite mean. Then* $X + \mathbb{E}[X] \le_{cx} 2X$.

Proof. It suffices to use the proposition 1.

Corollary 5. *Let* $X_1, X_2, ..X_n$ *and* Y *be* $(n+1)$ *random variables. If* $X_i \le_{cx} Y, i = 1, \dots, n$, *then* $\sum_{i=1}^{n} a_i X_i \le_{cx} Y$, *whenever* $a_i \ge 0, i = 1, \dots, n$ *and* $\sum_{i=1}^{n} a_i = 1$.

Proof. It suffices to use the lemma 1.

Lemma 2. *Let* X_1 *and* X_2 *be a pair of independent random variables and let* Y_1 *and* Y_2 *be another pair of independent random variables. If* $X_i \le_{cx} Y_i, i = 1,2$, *then* $X_1 X_2 \le_{cx} Y_1 Y_2$.

Proof. See [3]

Lemma 3. *Let* $X \le_{cx} Y$ *if and only* $\mathbb{E}[\upsilon(X)] \le \mathbb{E}[\upsilon(Y)]$ *for all convex function* υ, *provided expectation exit.*

Proof. See [8]

Corollary 6. *Let* X, Y *and* Z *be random variables such that* $X \le_{cx} Y$ *and* $Y \le_{cx} Z$, *then* $X \le_{cx} Z$.

Proof. It suffices to use the proposition 1.

Lemma 4. *Let* $X \in \mathbb{R}_+^n$ *and* $X_1 \le_{lr} \dots \le_{lr} X_n$ *are mutually independent. If* ***b*** *is weakly majorized by* ***a*** *(denoted by* $\boldsymbol{b} \prec\prec \boldsymbol{a}$*) and* $\boldsymbol{a} \in \mathbb{R}^n$, *then* $\sum_{i=1}^{n} b_i X_i \le_{icx} \sum_{i=1}^{n} a_i X_i$.

Proof. See [6]

Convex bounds for S_N

The main results of this paper are the following theorem, proposition and lemmas. Also, In this section we need the comonotonic notion.

Definition 8. A subset $A \in \mathbb{R}^n$ is said to be comonotonic if whenever $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ are elements of A, either $x_i \le y_i$ for all i or $y_i \le x_i$ for all i. A random vector $\mathbf{X} = (X_1, \dots, X_n) \in \mathbb{R}^n$ is said to be comonotonic if there is a comonotonic subset A of $\mathbb{R}^n$ such that $\mathrm{P}(\mathbf{X} \in \mathrm{A}) = 1$.

Lemma 5. *The following statements are equivalent:*

1) *The random vector $\boldsymbol{X} = (X_1, \dots, X_n)$ is comonotonic.*
2) *A random variable Z and non-decreasing function $f_i, i = 1, \dots, n$ exist such that*

$$\mathbf{X} \stackrel{\text{def}}{=} \left(f_1(Z), \dots, f_n(Z)\right).$$

This lemma implies that Comonotonicity is preserved under a non-decreasing transform on each component of $\boldsymbol{X}$.

Let $\mathrm{F}_1, \dots, \mathrm{F}_n$ be n univariate distribution functions. We use $\mathcal{R}(\mathrm{F}_1, \dots, \mathrm{F}_n)$ to denote the Fréchet space of all the n-dimensional random vectors whose marginal distributions are $\mathrm{F}_1, \dots, \mathrm{F}_n$, respectively.

Lemma 6. (Dhaene et al. (2002)) *If $\left(\tilde{X}_1, \dots, \tilde{X}_n\right) \in \mathcal{R}(F_1, \dots, F_n)$ is comonotonic, then*

$X_1 + \cdots + X_n \le_{cx} \tilde{X}_1 + \cdots + \tilde{X}_n$

for any $(X_1, \dots, X_n) \in \mathcal{R}(F_1, \dots, F_n)$.

Many problems in risk theory involve sums of r.v.'s: $S = X_1 + \cdots + X_n$ or $S_N = X_1 f(Y_1) + X_2 f(Y_2) + \cdots + X_n f(Y_n)$ (our model), in this subsection, we give convex upper and lower bound for S_N.

Theorem 1. *We note that*

$\tilde{S}_N = \tilde{X}_1 f(Y_1) + \tilde{X}_2 f(Y_2) + \cdots + \tilde{X}_n f(Y_n)$

For any random vector $X = (X_1, \dots, X_n)$ and $f(Y_i), i = 1, \dots, n$ we have

$$S_N \preccurlyeq_{lorenz} \tilde{S}_N .$$

Proof. It is suffices to prove stop-loss order because $\mathbb{E}(S_N) = \mathbb{E}(\tilde{S}_N)$. Hence, we have to prove that

$\mathbb{E}[(S_N - d)_+] \le \mathbb{E}[(\tilde{S}_N - d)_+]$

The following holds for all $(X_1 f(Y_1), X_2 f(Y_2), \dots, X_n f(Y_n))$ when $d_1 + d_2 + \dots + d_n = d$:

$$(X_1 f(Y_1) + X_2 f(Y_2) + \cdots + X_n f(Y_n) - d)_+$$
$$= (X_1 f(Y_1) - d_1 + X_2 f(Y_2) - d_2 + \cdots + X_n f(Y_n) - d_n)_+$$
$$\le ((X_1 f(Y_1) - d_1)_+ + (X_2 f(Y_2) - d_2)_+ + \cdots + (X_n f(Y_n) - d_n)_+)_+$$
$$= (X_1 f(Y_1) - d_1)_+ + (X_2 f(Y_2) - d_2)_+ + \dots + (X_n f(Y_n) - d)_+$$

Now taking expectations, we get that

$\mathbb{E}[(X_1 f(Y_1) + X_2 f(Y_2) + \dots + X_n f(Y_n) - d)_+] \le \sum_{i=1}^{n} \mathbb{E}[(X_i f(Y_i) - d_i)_+]$.

According to [3] we have

$\mathbb{E}[(S_N - d)_+] \leq \mathbb{E}[(X_i f(Y_i) - d_i)_+]$.
and according to corollary 1, we obtain
$S_N \preccurlyeq_{lorenz} \tilde{S}_N$.

Corollary 7. *If $f(Y_i) \leq 1, i = 1, \ldots, n,$ we can check easily that*

$$S_N \leq_{cx} \tilde{S}_N \leq_{cx} S \leq_{cx} \tilde{S}.$$

Remark 3. *For any random vector $X = (X_1, \ldots, X_n)$ and $f(Y_i) \leq 1, i = 1, \ldots, n$ we have*

$$S_N \preccurlyeq_{lorenz} \tilde{S}_N .$$

Application: Policy Limits and Deductible
We consider for the following model:
$S_N = X_1 f(Y_1) + X_2 f(Y_2) + \cdots + X_n f(Y_n)$ (M1)
where: $Y_i = \delta_i T_i$, S_N is total discounted loss, X_i are loss due to the *i*-th risk, T_i are time of occurrence of *i*-th insured risk and δ_i are discount rate capture the impact of financial environment. (X_i, T_i are independent non-negative random variables and δ_i are non-random numbers). Also, we will make the following assumptions:

1. $f(Y_i) \geq 0$; $\forall y_i$ and $\lim_{y_i \to +\infty} f(Y_i) = 0$.

2. $f(Y_i)$ is decreasing and convex function.

3. $Y_1, Y_2, \ldots, Y_n$ are mutually independent.

4. A policyholder exposed to risks $X_1, X_2, \ldots, X_n$ is granted a total of l dollars $(l > 0)$ as the policy limit with which (s)he can allocate arbitrarily among the *n* risks.

Remark 4. A very good property of the model (M1) is that X_i's characterize the scales of the losses while $f(Y_i)$ characterize the chances of the losses.
In this situation, if some risk occurs, the insurer will make the payment right after the event of the loss and the insurance coverage for this risk will terminate. However the insurance coverage for the other risks is still in effect. If $(l_1, \ldots, l_n)$ are the allocated policies we have $\forall i : l_i \geq 0$: and $\sum_{i=1}^{n} l_i = l$.When l is *n*-tuple admissible and $\mathcal{A}_n(l)$ denote the class of all such *n*-tuples. If $\mathbf{l} = (l_1, \ldots, l_n) \in \mathcal{A}_n(l)$ is chosen, then the discounted value of benefits obtained from the insurer would be

$$\sum_{i=1}^{n} (X_i \wedge l_i) f(Y_i)$$

If we take expected utility of wealth as the criterion for the optimal allocation, then the problem of the optimal allocation of policy limits is

$$Problem\ L: \max_{\mathbf{l} \in \mathcal{A}_n(l)} \mathbb{E}\left[u\left(w - \sum_{i=1}^{n} [X_i - (X_i \wedge l_i)] f(Y_i)\right)\right]$$

Where u is the utility function of the policyholder and w is the wealth (after premium).
Similarly, instead of policy limits, the policyholder may be granted a total of d dollars $(d > 0)$as the policy deductible with which (s) he can allocate arbitrarily among the n risks. If $\mathbf{d} = (d_1, \ldots, d_n) \in \mathcal{A}_n(d)$ are the allocated deductibles, $\forall i : d_i \geq 0, \sum_{i=1}^{n} d_i = d$,, and the discounted value of benefits obtained from the insurer would be

$$\sum_{i=1}^{n} (X_i - d_i)_+ f(Y_i)$$

Then the problem of the optimal allocation of policy deductibles is

$$Problem\ D: \max_{\mathbf{d} \in \mathcal{A}_n(d)} \mathbb{E}\left[u\left(w - \sum_{i=1}^{n} [X_i - (X_i - d_i)_+] f(Y_i)\right)\right]$$

Lemma 7. *If* $(\tilde{X}_1, \ldots, \tilde{X}_n) \in \mathcal{R}$ *comonotonic, then*

$$\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} (X_i - l_i)_+ f(Y_i)\right)\right] \leq \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} (\tilde{X}_i - l_i)_+ f(Y_i)\right)\right]$$

for any $(l_1, \ldots, l_n) \in \mathcal{A}_n(l)$ *and* $(X_1, \ldots, X_n) \in \mathcal{R}$ *independent of* $\mathbf{Y}$.

Proof. See [19]

Lemma 8. *Suppose that* $\tilde{u}$ *is increasing and convex. If* $(\tilde{X}_1, \ldots, \tilde{X}_n) \in \mathcal{R}$ *is comonotonic and independent of* $\mathbf{Y}$, *then*

$$\mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} (X_i \wedge d_i)_+ f(Y_i)\right)\right] \leq \mathbb{E}\left[\tilde{u}\left(\sum_{i=1}^{n} (\tilde{X}_i \wedge d_i)_+ f(Y_i)\right)\right]$$

Proof. See [20]

Remark 5. *We can take* $f(Y_i) = e^{-Y_i}$ *for simplicity.*

Some examples

Individual and collective risk model

The classical individual and collective model of risk theory has the form, $X_{Ind} = \sum_{i=1}^{n} b_i I_i$, $X_{Coll} = \sum_{i=1}^{n} b_i N_i$, where $I_i \backsim Bernoulli(1, p_i)$ and $N_i \backsim poisson(\lambda_i)$.
With probability p_i contract i will yield a claim of size $b_i \geq 0$ for any of the n policies.
As an application of stochastic and stop loss ordering we get that the collective risk model X_{Coll} leads to an overestimate of the risks and, therefore, also to an increase of the corresponding risk premiums for the whole portfolio $X_{Ind} \leq_{lorenz} X_{Coll}$.

Reinsurance contracts

We consider reinsurance contracts $I(X)$ for a risk X, where $0 \leq I(X) \leq X$ is the reinsured part of the risk X and $X - I(X)$ is the retained risk of the insurer. Consider the stop loss reinsurance contract $I_a(X) = (X - a)_{+}$, where a is chosen such that $EI_a(X) = EI(X)$. Then for any reinsurance contract $I(X)X - I_a(X) \leq_{lorenz} X - I(X)$.

Dependent portfolios increase risk

Let $Y_i = \sum_{i=1}^{m} \alpha_i X_i$, where α_i and $X_i \backsim Bernoulli$ with $\sum_{i=1}^{m} \alpha_i = 1$, then $Y_i \backsim Bernoulli$. It is interesting to compare the total risk $T_n = \sum_{i=1}^{n} Y_i$ in the mixed model (X_i) with the total risk $S_n = \sum_{i=1}^{n} W_i$ in an independent portfolio model (W_i), where $W_i \backsim Bernoulli$ are distributed identical to X_i. Then we obtain

$S_n \leq_{lorenz} T_n$

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