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Polar Reflexivity In Locally Convex Spaces

Atarsingh Meena[1] and G. C. Dubey[2]

Department of Mathematics,
Govt. M.G.M. Post-Graduate College, Itarsi (M.P.), India. Pin code 461111
Email: ameenaprinc@gmail.com [1] gcd.1951@gmail.com [2]

Abstract

In this paper, we investigate that a metrizable locally convex space is polar semi-reflexive if and only if it is polar reflexive. We also examine that the quotient space of polar semi-reflexive locally convex space is not polar semi-reflexive, in general. It is also discussed that the strong dual of polar reflexive locally convex space is polar semi-reflexive.

Key words: Uniform convergence, reflexive, polar reflexive, p-determined space, k-space.

AMS 2010 Subject Classification: 46A25

1 INTRODUCTION

For a locally convex space $E[\tau]$, which we always consider Hausdorff, the dual is denoted as E'. The strong dual of $E[\tau]$ is $E'[\tau_b(E)]$ and the bidual of $E[\tau]$ is $E''= (E'[\tau_b(E)])'$. If $E''= E$, then $E[\tau]$ is called semi-reflexive. A semi-reflexive locally convex space $E[\tau]$ is called reflexive provided $\tau = \tau_b(E')$.

We follow the notion of Köthe [1] for notations and terminology, unless specifically mentioned.

The topology on E' of uniform convergence over the class of τ-precompact sets (in E)is denoted by τ^0. We have $\tau^0 \leq \tau_b(E)$. The topology on $(E'[\tau^0])'$ of uniform convergence over the class of τ^0 -precompact subsets of $E'[\tau^0]$ is denoted as τ^{00}.

Following [1], we define polar semi-reflexive and polar reflexive locally convex spaces:

1.1. Definition: A locally convex space $E[\tau]$ is called polar semi-reflexive if $E = (E'[\tau^0])'$. Polar semi-reflexive space $E[\tau]$ is called polar reflexive if $\tau = \tau^{00}$ i.e. $(\tau^0)^{o.}$

1.2. From [1], it is known that if a locally convex space is semi-reflexive (resp. reflexive), then it is polar semi-reflexive (resp. polar reflexive).

1.3. Brauner [2] has discussed p-complete and p-reflexive locally convex spaces. A Locally convex space $E[\tau]$ is called p-complete if every closed precompact subset of $E[\tau]$ is complete. A Locally convex space $E[\tau]$ is called p-reflexive if the evaluation map from E to $(E'[\tau^0])'[\tau^{00}]$ is a topological isomorphism. We note, firstly, that p-reflexive space, by definition, is nothing but polar reflexive space. Secondly, $E[\tau]$ is p-complete means closures of precompact sets are compact (and so $\tau_s(E')$-compact). It is equivalent to $\tau_s(E) \leq \tau^0 \leq \tau_k(E)$. i.e. $E = (E'[\tau^0])'$. Therefore, p-complete space is same as polar semi-reflexive space. Thus p-complete and p-reflexive spaces are nothing but polar semi-reflexive and polar reflexive spaces, respectively.

Recall that a closed, absorbent, absolutely convex subset of a locally convex space $E[\tau]$ is called a barrel. A barrel B in $E[\tau]$ is called a p-barrel if for each precompact subset P of E which contains **0** (zero)**,** $B \cap P$ is a neighborhood of **0** in P for the relative topology induced from $E[\tau]$. Locally convex space $E[\tau]$ is called p-determined if every p-barrel in $E[\tau]$ is a neighborhood of **0.**

1.4. A locally convex space $E[\tau]$ is polar reflexive if and only if it is both polar semi-reflexive and p-determined ([2], proposition 0.5(a)) This characterization is also discussed in [3](Theorem-5) and [4](Theorem-3).

2. RESULTS

2.1. Theorem: A barreled locally convex space is polar semi-reflexive if and only if it is polar reflexive.

Proof: In a barreled locally convex space $E[\tau]$, every barrel is a neighborhood of **0**. So $E[\tau]$ is always p-determined. Therefore, if $E[\tau]$ is polar semi-reflexive, it becomes polar reflexive. Conversely, polar reflexive locally convex space is, by definition, always polar semi-reflexive.

If a subset M of a locally convex space $E[\tau]$ is closed whenever $M \cap C$ is closed for all compact subsets C of E, then $E[\tau]$ is called a k-space (see [2]). Now, for metrizable locally convex spaces, we have the following result:

2.2. Theorem: A metrizable locally convex space $E[\tau]$ is a k-space.

Proof*:* Let M be a subset of $E[\tau]$ such that $M \cap C$ is closed for all compact subsets C of E. Let x be a closure point of M. Since $E[\tau]$ is metrizable, we can have a sequence (x_n) of points of M such that (x_n) converges to x ([1], §3, 5(1)). Take S as the set consists of x and all the terms x_n of the sequence. It is clear that S is compact. Therefore, by assumption, $M \cap S$ is closed. So $x \in M \cap S$ and so $x \in M$. Hence $E[\tau]$ is a k-space.

Now we have the following assertion-

2.3. Theorem: A metrizable locally convex space $E[\tau]$ is polar semi reflexive if and only if it is polar reflexive.

Proof: Let $E[\tau]$ be a metrizable locally convex space which is also polar semi reflexive. $E[\tau]$ is k-space, by theorem 2.2. Since a locally convex k-space is p-determined ([2], proposition 0.8), so $E[\tau]$ is p-determined. Hence $E[\tau]$ is polar reflexive, by[2], proposition 0.5(a). Converse of the theorem is trivial.
Brauner [2] has discussed, in proposition 0.5(c), that every closed subspace of a polar semi-reflexive locally convex space is polar semi-reflexive. Let us discuss the case of quotient spaces. We note the fact that quotient space of a semi-reflexive (reflexive) locally convex space by a closed subspace, in general, is not semi-reflexive (See [1], §23, 5). This fact enables us to assert a result for polar semi-reflexive spaces in the next theorem which we establish with the following two lemmas-

2.4. Lemma: A locally convex space $E[\tau]$ is semi-reflexive if and only if $E[\tau_s(E')]$ is semi-reflexive.

Proof: All the locally convex topologies on E compatible for the dual pair (E′, E) define the same collection of bounded sets. Therefore, in particular, $E[\tau]$ and $E[\tau_s(E')]$ have identical strong duals as well as biduals. Hence $E[\tau]$ is semi-reflexive if and only if $E[\tau_s(E')]$ is semi-reflexive.

2.5. Lemma: A locally convex space equipped with the weak topology is semi-reflexive if and only if it is polar semi-reflexive (with weak topology).

Proof: Let $E[\tau]$ be a locally convex space with the dual E′. We know that the weakly bounded subsets of $E[\tau]$ are the same as the weakly precompact subsets of E. Therefore, $(E'[\tau_b(E)])' = (E'[\ (\tau_s(E')\ )^o])'$. So $E[\tau_s(E')]$ is semi-reflexive whenever it is polar semi-reflexive. Conversely, a semi-reflexive locally convex space is always polar semi-reflexive ([1], §23, 9(2)).
Recall the fact that if $E[\tau]$ is a locally convex space and F is a closed subspace of $E[\tau]$, then the weak topology for the quotient space E/F is the quotient topology derived from the weak topology of E. Also note that a closed subspace in the original topology τ is also closed in the weak topology. Now we have the following-

2.6. Theorem: The quotient space of a polar semi-reflexive locally convex space by a closed subspace is not polar semi-reflexive, in general.

Proof: We consider a locally convex space $E[\tau]$ and a closed subspace F such that $E[\tau]$ is semi-reflexive but the quotient E/F is not semi-reflexive (for existence see [1], §23, 5). By lemma-2.4 $E[(\tau_s(E')]$ is semi-reflexive and the quotient space E/F, equipped with weak topology, is not semi-reflexive. Hence by lemma-2.5, $E[(\tau_s(E')]$ is polar semi-reflexive and the quotient space E/F, equipped with weak topology, is not polar semi-reflexive.
However, in particular, we have

2.7. Theorem: The quotient space of an (F)-space by a closed subspace is polar reflexive.

Proof: If E is an (F)-space and F is a closed subspace of E, then the quotient space E/F is again an (F)-space by [1], §18, 3(4). Therefore, being an (F)- space, the quotient space E/F is polar reflexive ([1], §23, 9(5))..
Now we discuss polar reflexivity in the dual space E′, considered with the topologies τ^0 and $\tau_b(E)$, respectively.
By [2], proposition 0.5(b), if E[τ] is polar reflexive, then the dual E′[τ^0] is also polar reflexive. For the strong dual we have the following assertion:

2.8. Theorem: If E[τ] is polar reflexive, then the strong dual E′[τ_b(E)] is polar semi-reflexive.

Proof: Let E[τ] be polar reflexive. That is, (E′[τ^0])′ = E and $\tau = \tau^{oo}$. To prove that E′[τ_b(E)] is polar semi-reflexive we show that (E″[(τ_b(E) $)^o$])′ = E′. We know that $\tau^o \leq \tau_b(E)$. Therefore, if a subset A in E′ is τ_b(E)- precompact then it is τ^o-precompact and so τ^{oo}–equicontinuous i.e. τ-equicontinuous. Therefore, Ã (τ_s(E)- closure of A) is τ_s(E)-compact([1], §21, 3(1), §20,9(4)). So Ã is τ_b(E)-complete. It means A is τ_b(E)-relatively compact. Hence ($\boldsymbol{\tau_b}$(E) $)^0 \leq \boldsymbol{\tau_k}$(E′, E″) on E″ which implies (E″[(τ_b(E) $)^o$])′= E′. This completes the proof.

Note: It is not known whether the strong dual of polar reflexive locally convex space can be polar reflexive.

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