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q-Bernoulli, q-Euler and mixed-type polynomials

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Abstract

In this paper, we consider q-analogues of Bernoulli, Euler and mixed-type polynomials. From those polynomials, we derive some new identities.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper Z_p, Q_p and C_p will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of Q_p. Let ν_p be the normalized exponential valuation of C_p with $|p|_p = p^{-\nu_p(p)} = 1/p$. Let us assume that q is an indeterminate in C_p with $|1-q|_p < p^{-1/(p-1)}$. The q-number of x is defined as $[x]_q = (1-q^x)/(1-q)$. Note that $\lim_{q\to 1}[x]_q = x$. Let $UD(Z_p)$ be the space of uniformly differentiable functions on Z_p. For $f \in UD(Z_p)$, the bosonic p-adic q-integral on Z_p is defined by Kim to be

$$I_q(f) = \int_{Z_p} f(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x. \tag{1.1}$$

Thus, by (1.1), we get

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f^{'}(0), \tag{1.2}$$

where $f_1(x) = f(x+1)$, [15, 16].

In [16], Kim also defined the fermionic p-adic q-integral on Z_p as follows:

$$I_{-q}(f) = \lim_{N\to\infty} \frac{1+q}{1+q^{p^N}} \sum_{x=0}^{p^N-1} f(x)(-1)^x q^x. \tag{1.3}$$

Thus, by (1.3), we get

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0),$$

where $f_1(x) = f(x+1)$.

Let

$$I_0(f) = \lim_{q\to 1} I_q(f) = \int_{Z_p} f(x)d\mu_0(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\frac{1}{p^N} \tag{1.4}$$

and

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{Z_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{1.5}$$

From (1.4) and (1.5), we have

$$I_0(f_1) - I_0(f) = f^{'}(0), I_{-1}(f_1) + I_{-1}(f) = 2f(0). \tag{1.6}$$

By (1.6), we get

$$\int_{Z_p} e^{(x+y)t} d\mu_0(y) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \tag{1.7}$$

and

$$\int_{Z_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.8}$$

where $B_n(x)$ and $E_n(x)$ are respectively Bernoulli and Euler polynomials, (see [1, 2, 3, 4, 5, 6, 12, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 24, 25, 22, 23]). When $x = 0$, $B_n = B_n(0)$ are called Bernoulli numbers and $E_n = E_n(0)$ are said to be Euler numbers. As is known, the q-Daehee polynomials are given by

$$\left(\frac{q - 1 + \frac{q-1}{\log q} \log(1+t)}{qt + q - 1} \right) (1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{n!}. \tag{1.9}$$

When $x = 0$, $D_{n,q} = D_{n,q}(0)$ are called the q-Daehee numbers, (see [7, 25]). It is known that the q-Changhee polynomials are defined by

$$\left(\frac{[2]_q}{qt + [2]_q} \right) (1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!}, \quad \text{(see [17])}.$$

When $x = 0$, $Ch_{n,q} = Ch_{n,q}(0)$ are called the q-Changhee numbers.

In the viewpoint of (1.7) and (1.8), we consider q-analogues of Bernoulli, Euler and mixed-type polynomials. From those polynomials, we derive new and interesting identities.

2. q-analogue of Bernoulli, Euler and mixed-type polynomials

For $r \in N$, the higher-order q-Bernoulli polynomials are defined by the generating function to be

$$\begin{aligned} \int_{Z_p} \cdots \int_{Z_p} e^{(x+y_1+\cdots+y_r)t} d\mu_q(y_1) \cdots d\mu_q(y_r) &= \left(\frac{q - 1 + \frac{q-1}{\log q} t}{qe^t - 1} \right)^r e^{xt} \\ &= \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.1}$$

When $x = 0$, $B_{n,q}^{(r)} = B_{n,q}^{(r)}(0)$ are called the higher-order q-Bernoulli numbers. For $r = 1$, $B_{n,q}(x) = B_{n,q}^{(1)}(x)$ are called the q-Bernoulli polynomials. When $x = 0$, $B_{n,q} = B_{n,q}^{(1)}(0)$ are said to be the q-Bernoulli numbers.

From (2.1), we can derive the following relation:

$$(qB_q + 1)^n - B_{n,q} = \begin{cases} q - 1, & \text{if } n = 0 \\ \dfrac{q-1}{\log q}, & \text{if } n = 1 \\ 0, & \text{if } n > 1 \end{cases} \tag{2.2}$$

with the usual convention about replacing B_q^n by $B_{n,q}$.

By (2.1), we easily get

$$B_{n,q}^{(r)}(x)=\sum_{l=0}^{n}\binom{n}{l}B_{l,q}^{(r)}x^{n-l}. \tag{2.3}$$

Now, we consider the higher-order q-Euler polynomials as follows:

$$\left(\frac{[2]_q}{qe^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}E_{n,q}^{(r)}(x)\frac{t^n}{n!}. \tag{2.4}$$

When $x=0$, $E_{n,q}^{(r)}=E_{n,q}^{(r)}(0)$ are called the higher-order q-Euler numbers. For $r=1$, $E_{n,q}(x)=E_{n,q}^{(1)}(x)$ are called the q-Euler polynomials. When $x=0$, $E_{n,q}=E_{n,q}(0)$ are said to be the q-Euler numbers.

From (2.4), we have

$$q(E_q+1)^n+E_{n,q}=[2]_q\delta_{0,n}$$

with the usual convention about replacing E_q^n by $E_{n,q}$.

By (2.4), we easily get

$$E_{n,q}^{(r)}(x)=\sum_{l=0}^{n}\binom{n}{l}E_{l,q}^{(r)}x^{n-l}. \tag{2.5}$$

The higher-order q-Daehee polynomials are also defined by the generating function to be

$$\left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{qt+q-1}\right)^r(1+t)^x=\sum_{n=0}^{\infty}D_{n,q}^{(r)}(x)\frac{t^n}{n!}. \tag{2.6}$$

When $x=0$, $D_{n,q}^{(r)}=D_{n,q}^{(r)}(0)$ are called the higher-order q-Daehee numbers.

Finally, we consider the higher-order q-Changhee polynomials as follows:

$$\left(\frac{[2]_q}{qt+[2]_q}\right)^r(1+t)^x=\sum_{n=0}^{\infty}Ch_{n,q}^{(r)}(x)\frac{t^n}{n!}. \tag{2.7}$$

When $x=0$, $Ch_{n,q}^{(r)}=Ch_{n,q}^{(r)}(0)$ are said to be the higher-order q-Changhee numbers.

In this section, we assume that $t?C_p$ with $|t|_p<p^{-1/(p-1)}$. Then we have

$$\begin{aligned}\int_{Z_p}\cdots\int_{Z_p}&(1+t)^{x_1+\cdots+x_r+x}d\mu_q(x_1)\cdots d\mu_q(x_r) \qquad (2.8)\\ &=\left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{qt+q-1}\right)^r(1+t)^x\\ &=\sum_{n=0}^{\infty}D_{n,q}^{(r)}(x)\frac{t^n}{n!}.\end{aligned}$$

Now, we observe that

$$\left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{qt+q-1}\right)^r(1+t)^x = \left(\frac{q-1+\frac{q-1}{\log q}\log(1+t)}{qe^{\log(1+t)}-1}\right)^r e^{x log(1+t)} \tag{2.9}$$
$$= \sum_{m=0}^{\infty} B_{m,q}^{(r)}(x)\frac{1}{m!}(\log(1+t))^m$$
$$= \sum_{m=0}^{\infty} B_{m,q}^{(r)}(x)\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n} B_{m,q}^{(r)}(x)S_1(n,m)\right\}\frac{t^n}{n!},$$

where $S_1(n,m)$ is the Stirling number of the first kind.

Therefore, by (2.8) and (2.9), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$\int_{Z_p}\cdots\int_{Z_p}\binom{x_1+\cdots+x_r+x}{n}d\mu_q(x_1)\cdots d\mu_q(x_r) = \frac{D_{n,q}^{(r)}(x)}{n!}$$
$$= \frac{1}{n!}\sum_{m=0}^{n} B_{m,q}^{(r)}(x)S_1(n,m).$$

From (1.3), we can also derive

$$\int_{Z_p}\cdots\int_{Z_p}(1+t)^{x_1+\cdots+x_r+x}d\mu_{-q}(x_1)\cdots d\mu_{-q}(x_r) = \left(\frac{[2]_q}{qt+[2]_q}\right)^r(1+t)^x \tag{2.10}$$
$$= \sum_{n=0}^{\infty} Ch_{n,q}^{(r)}(x)\frac{t^n}{n!}.$$

Note that

$$\left(\frac{[2]_q}{qt+[2]_q}\right)^r(1+t)^x = \left(\frac{[2]_q}{qe^{\log(1+t)}+1}\right)^r e^{x\log(1+t)} \tag{2.11}$$
$$= \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x)\frac{(\log(1+t))^n}{n!}$$
$$= \sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m} E_{n,q}^{(r)}(x)S_1(m,n)\right\}\frac{t^m}{m!}.$$

Thus, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.2. For $m \geq 0$, we have

$$Ch_{m,q}^{(r)}(x) = \sum_{n=0}^{m} E_{n,q}^{(r)}(x) S_1(m,n).$$

Let us define the following polynomials:

$$T_{n,q}^{(r,s)}(x) = \int_{Z_p} \cdots \int_{Z_p} E_{n,q}^{(s)}(x + y_1 + \cdots + y_r) d\mu_q(y_1) \cdots d\mu_q(y_r). \tag{2.12}$$

Then, the generating function of $T_{n,q}^{(r,s)}(x)$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(r,s)}(x)\frac{t^n}{n!} &= \int_{Z_p} \cdots \int_{Z_p} \sum_{n=0}^{\infty} E_{n,q}^{(s)}(x + y_1 + \cdots + y_r)\frac{t^n}{n!} d\mu_q(y_1) \cdots d\mu_q(y_r) \\ &= \left(\frac{[2]_q}{qe^t + 1}\right)^s \int_{Z_p} \cdots \int_{Z_p} e^{(x+y_1+\cdots+y_r)t} d\mu_q(y_1) \cdots d\mu_q(y_r) \\ &= \left(\frac{[2]_q}{qe^t + 1}\right)^s \left(\frac{q - 1 + \frac{q-1}{\log q} t}{qe^t - 1}\right)^r e^{xt} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{l=0}^{n} \binom{n}{l} E_{l,q}^{(s)} B_{n-l,q}^{(r)}(x) \right\} \frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, we have

$$T_{n,q}^{(r,s)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{l,q}^{(s)} B_{n-l,q}^{(r)}(x).$$

By replacing t by $\log(1+t)$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(r,s)}(x)\frac{(\log(1+t))^n}{n!} &= \left(\frac{[2]_q}{qt + [2]_q}\right)^s \left(\frac{q - 1 + \frac{q-1}{\log q}\log(1+t)}{qt + q - 1}\right)^r (1+t)^x \\ &= \left(\sum_{l=0}^{\infty} Ch_{l,q}^{(s)} \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} D_{m,q}^{(r)}(x)\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{n}{m} D_{m,q}^{(r)}(x) Ch_{n-m,q}^{(s)} \right\} \frac{t^n}{n!}.\end{aligned} \tag{2.14}$$

We observe that

$$\sum_{m=0}^{\infty} T_{m,q}^{(r,s)}(x)\frac{(\log(1+t))^m}{m!} = \sum_{m=0}^{\infty} T_{m,q}^{(r,s)}(x)\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n} T_{m,q}^{(r,s)}(x)S_1(n,m)\right\}\frac{t^n}{n!}. \tag{2.15}$$

Therefore, by (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. For $n \geq 0$, we have

$$\sum_{m=0}^{n} T_{m,q}^{(r,s)}(x)S_1(n,m) = \sum_{m=0}^{n}\binom{n}{m} D_{m,q}^{(r)}(x)Ch_{n-m,q}^{(s)}.$$

Corollary 2.5. For $n \geq 0$, we have

$$\sum_{m=0}^{n}\{\sum_{l=0}^{m}\binom{m}{l}E_{l,q}^{(s)}B_{m-l,q}^{(r)}(x)\}S_1(n,m) = \sum_{m=0}^{n}\binom{n}{m}D_{m,q}^{(r)}(x)Ch_{n-m,q}^{(s)}.$$

From (1.5), we can derive

$$\sum_{n=0}^{\infty} D_{n,q}^{(r)}(x)\frac{(e^t-1)^n}{n!} = \left(\frac{q-1+\frac{q-1}{\log q}t}{qe^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x)\frac{t^n}{n!}. \tag{2.16}$$

Note that

$$\sum_{n=0}^{\infty} D_{n,q}^{(r)}(x)\frac{1}{n!}(e^t-1)^n = \sum_{n=0}^{\infty} D_{n,q}^{(r)}(x)\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m} D_{n,q}^{(r)}(x)S_2(m,n)\right\}\frac{t^m}{m!}. \tag{2.17}$$

Thus, by (2.16) and (2.17), we obtain the following theorem.

Theorem 2.6. For $m \geq 0$, we have

$$B_{m,q}^{(r)}(x) = \sum_{n=0}^{m} D_{n,q}^{(r)}(x)S_2(m,n),$$

where $S_2(m,n)$ is the Stirling number of the second kind.

From (2.7), we have

$$E_{m,q}^{(r)}(x) = \sum_{n=0}^{m} Ch_{n,q}^{(r)}(x) S_2(m,n).$$

Let us consider the higher-order q-Daehee-Changhee mixed-type polynomials as follows:

$$DC_{n,q}^{(r,s)}(x) = \int_{Z_p} \cdots \int_{Z_p} D_{n,q}^{(r)}(x + y_1 + \cdots + y_s) d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_s), \tag{2.18}$$

where $n \geq 0$.

From (2.18), we can derive the generating function of $DC_{n,q}^{(r,s)}(x)$ as follows:

$$\sum_{n=0}^{\infty} DC_{n,q}^{(r,s)}(x) \frac{t^n}{n!} = \left(\frac{q - 1 + \frac{q-1}{\log q} \log(1+t)}{qt + q - 1} \right)^r \left(\frac{[2]_q}{qt + [2]_q} \right)^s (1+t)^x \tag{2.19}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{n}{m} D_{m,q}^{(r)}(x) Ch_{n-m,q}^{(s)} \right\} \frac{t^n}{n!}.$$

Therefore, by (2.19), we obtain the following theorem.

Theorem 2.7. For $n \geq 0$, we have

$$DC_{n,q}^{(r,s)}(x) = \sum_{m=0}^{n} \binom{n}{m} D_{m,q}^{(r)}(x) Ch_{n-m,q}^{(s)}.$$

By replacing t by $e^t - 1$ in (2.19), we get

$$\sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{n}{m} D_{m,q}^{(r)}(x) Ch_{n-m,q}^{(s)} \right\} \frac{(e^t-1)^n}{n!} = \left(\frac{q - 1 + \frac{q-1}{\log q} t}{qe^t - 1} \right)^r \left(\frac{[2]_q}{qe^t + 1} \right)^s e^{xt} \tag{2.20}$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} B_{l,q}^{(r)} E_{n-l,q}^{(s)}(x) \frac{t^n}{n!}.$$

We observe that

$$\sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{n}{m} D_{m,q}^{(r)}(x) Ch_{n-m,q}^{(s)} \right\} \frac{(e^t-1)^n}{n!} \tag{2.21}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{n}{m} D_{m,q}^{(r)}(x) Ch_{n-m,q}^{(s)} \right\} \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}$$

$$= \sum_{l=0}^{\infty} \left\{ \sum_{n=0}^{l} \sum_{m=0}^{n} \binom{n}{m} D_{m,q}^{(r)}(x) Ch_{n-m,q}^{(s)} S_2(l,n) \right\} \frac{t^l}{l!}.$$

Therefore, by (2.20) and (2.21), we obtain the following theorem.

Theorem 2.8. For $l \geq 0$, we have

$$\sum_{m=0}^{l} \binom{l}{m} B_{m,q}^{(r)} E_{l-m,q}^{(s)}(x) = \sum_{n=0}^{l} \sum_{m=0}^{n} \binom{n}{m} D_{m,q}^{(r)}(x) Ch_{n-m,q}^{(s)} S_2(l,n).$$

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