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Blow-up for Semidiscretizations of some Reaction-Diffusion Equations with a Nonlinear Convection Term

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Abstract

This paper concerns the study of the numerical approximation for the following parabolic equations with a nonlinear convection term

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) - u^q(x,t)u_x(x,t) + u^p(x,t), & 0 < x < 1, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x) > 0, & 0 \le x \le 1, \end{cases}$$

where $q \ge 1$ and $p \ge q+1$.

We obtain some conditions under which the solution of the semidiscrete form of the above problem blows up in a finite time and estimate its semidiscrete blow-up time. We also prove that the semidiscrete blow-up time converges to the real one, when the mesh size goes to zero. Finally, we give some numerical experiments to illustrate ours analysis.

AMS subject classifications(2000): 35K58, 35B44, 35B50, 35B51, 65M06, 35K28.

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1. Introduction

Consider the following boundary value problem

$$u_t(x,t) = u_{xx}(x,t) - u^q(x,t)u_x(x,t) + u^p(x,t), \quad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x) > 0, \quad 0 \le x \le 1, \tag{3}$$

where $q \ge 1$, $p \ge q+1$, $u_0 \in C^2([0,1])$, u_0 is nondecreasing on the interval $(0,1)$ and verifies

$$u_0^{'}(0) = 0, \; u_0^{'}(1) = 0, \tag{4}$$

$$u_0^{''}(x) - u^q(x)u_0^{'}(x) + u_0^p(x) \ge 0, \quad 0 \le x \le 1, \tag{5}$$

$$u_0(x) > -\frac{p(p-1)}{q} u_0^{'}(x), \quad 0 < x < 1, \tag{6}$$

1.1 Definition

We say that the solution u of (1)-(3) blows up in a finite time if there exists a finite time T_b such that $\|u(.,t)\|_\infty < \infty$ for $t \in [0, T_b)$ but

$\lim_{t \to T_b} \|u(.,t)\|_\infty = \infty$.

The time T_b is called the blow-up time of the solution u.

These equations arise in the theory of heat conduction. The heat transfer is the propagation of the heat from one place to another in a medium or between two different mediums. It is due to the movements of atoms and molecules in a material. Heat can be transferred between solids, liquids and gases or even in vaccum/space. Transfer of heat within a fluids is by conduction. The convection is the transfer of heat by the movement of fluids. The first equations is a heat equation including a nonlinear convection term $u^q u_x$ and a nonlinear source u^p. It is the term of convection which ensures the movement, generates instability and is also responsible of the turbulent appearance (here we'll refer to it as intermittent since we are in one dimension) when it happens (see [13], [19], [22], [23], [27]).

The theoretical study of blow-up solutions for the reaction-diffusion equations with a nonlinear convection term has been the subject of investigations of many authors (see [3], [6], [7], [8], [9],[23], [24], [25] and the references cited therein). Local in time existence and uniqueness of the solution have been proved(see [4], [5], [21], [28] and the references cited therein). Here, we are interesting in the numerical study using a semidiscrete form of (1)-(3). We give some assumptions under which the solution of a semidiscrete form of (1)-(3) blows up in a finite time and estimate its semidiscrete blow-up time. We also show that the semidiscrete blow-up time converges to the theoretical one when the mesh size goes to zero. A similar study has been undertaken (see [1], [22], [23]).

The paper is organized as follows. In the next section, we present a semidiscrete scheme of (1)-(3) and give some lemmas which will be used throughout the paper. In section 3, under some conditions, we prove that the solution of the semidiscrete form of (1)-(3) blows up in a finite time. In section 4, we study the convergence of the semidiscrete blow-up time. Finally, in last section, taking some discrete forms of (1)-(3), we give some numerical experiments.

2. Properties of the semidiscrete scheme

In this section, we give some lemmas which will be used later. We start by the construction of the semidiscrete scheme. Let I be a positive integer and let $h=1/I$. Define the grid $x_i = ih$, $0 \le i \le I$ and approximate the solution u of (1)-(3) by the solution $U_h(t) = (U_0(t), U_1(t), \cdots U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) - U_i^q(t)\delta^0 U_i(t) + U_i^p(t), \quad 1 \le i \le I-1, \quad t \in \left(0, T_b^h\right), \tag{7}$$

$$\frac{dU_0(t)}{dt} = \delta^2 U_0(t) + U_0^p(t), \quad t \in \left(0, T_b^h\right), \tag{8}$$

$$\frac{dU_I(t)}{dt} = \delta^2 U_I(t) + U_I^p(t), \quad t \in \left(0, T_b^h\right), \tag{9}$$

$$U_i(0) = \varphi_i > 0, \quad 0 \le i \le I, \tag{10}$$

where

$q \ge 1,\ p \ge q+1,$

$$\delta^2 U_i(t) = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{h^2},\ 1 \le i \le I-1,$$

$$\delta^2 U_0(t) = \frac{2U_1(t) - 2U_0(t)}{h^2}, \quad \delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2},$$

$$\delta^0 U_i(t) = \frac{U_{i+1}(t) - U_{i-1}(t)}{2h},\ 1 \le i \le I-1,$$

$$\delta^0 U_0(t) = 0, \quad \delta^0 U_I(t) = 0,$$

$$\delta^+ \varphi_i = \frac{\varphi_{i+1} - \varphi_i}{h},\ 0 \le i \le I-1,$$

$$\delta^+ \varphi_i \le 0,\ 0 \le i \le I-1,$$

$$\varphi_i^{p-1} > -\frac{p(p-1)}{q} h \varphi_{i-1}^{p-2} \delta^0 \varphi_i,\ 1 \le i \le I-1.$$

Here, $\left(0, T_b^h\right)$ is the maximal time interval on which $\|U_h(t)\|_\infty$ is finite, where $\|U_h(t)\|_\infty = \max_{0 \le i \le I} |U_i(t)|$.

When the time T_b^h is finite, we say that the solution $U_h(t)$ of (8)-(10) blows up in a finite time, and the time T_b^h is called the blow-up time of the solution $U_h(t)$.

Lemma 2.1 Let $a_h(t), b_h(t) \in C^0\left([0,T), \Re^{I+1}\right)$ and let $V_h(t) \in C^1\left([0,T), \Re^{I+1}\right)$ where $b_h(t)\delta^0 V_h(t) \le 0$, such that

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + b_i(t)\delta^0 V_i(t) + a_i(t) V_i(t) \ge 0, \quad 0 \le i \le I, \quad t \in (0,T), \tag{11}$$

$$V_i(0) \ge 0, \quad 0 \le i \le I. \tag{12}$$

Then we have

$$V_i(t) \ge 0,\ 0 \le i \le I,\ t \in (0,T). \tag{13}$$

Proof. Let T_0 be any quantity satisfying the inequality $T_0 < T$ and define the

vector $Z_h(t) = e^{\lambda t} V_h(t)$ where λ is such that
$a_i(t) - \lambda > 0 \ for \ 0 \le i \le I, \quad t \in [0, T_0].$

Let $m = \min_{0 \le i \le I, 0 \le t \le T_0} Z_i(t)$. Since, for $i \in \{0, \cdots, I\}$, $Z_i(t)$ is a continuous function on the compact $[0, T_0]$, there exists $i_0 \in \{0, \cdots, I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$. We observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \quad 0 \le i_0 \le I, \tag{14}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0, \ 1 \le i_0 \le I - 1, \tag{15}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \ge 0 \ \ if \ \ i_0 = 0, \tag{16}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \ge 0 \ \ if \ \ i_0 = I. \tag{17}$$

From (11), we obtain the following inequality

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + b_{i_0}(t_0)\delta^0 Z_{i_0}(t_0) + (a_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \ge 0. \tag{18}$$

It follows from (14)-(18) that

$$(a_{i_0}(t_0) - \lambda) Z_{i_0}(t_0) \ge 0, \tag{19}$$

which implies that $Z_{i_0}(t_0) \ge 0$ because $a_{i_0}(t_0) - \lambda > 0$.

We deduce that $V_h(t) \ge 0$ for $t \in [0, T_0]$ and the proof is complete.

Lemma 2.2 Let $V_h(t), W_h(t) \in C^1([0, T], \Re^{I+1})$ and $f \in C^1(\Re \times \Re, \Re)$ such that

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + V_i^q(t)\delta^0 V_i(t) + f(V_i(t), t) < \frac{dW_i(t)}{dt} -$$

$$\delta^2 W_i(t) + W_i^q(t)\delta^0 W_i(t) + f(W_i(t), t), \ 0 \le i \le I, \ t \in (0, T), \tag{20}$$

$$V_i(0) < W_i(0), \quad 0 \le i \le I. \tag{21}$$

Then, we have

$V_i(t) < W_i(t), \ 0 \le i \le I, \ t \in (0, T).$

Proof. Define the vector $Z_h(t) = W_h(t) - V_h(t)$. Let t_0 be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0), 0 \le i \le I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \cdots, I\}$.

We remark that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \quad 0 \le i_0 \le I,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0, \ 1 \le i_0 \le I - 1,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_1(t_0) - 2Z_0(t_0)}{h^2} \ge 0 \ \ if \ \ i_0 = 0,$$

$$\delta^2 Z_{i_0}(t_0) = \frac{2Z_{I-1}(t_0) - 2Z_I(t_0)}{h^2} \ge 0 \ \ if \ \ i_0 = I,$$

Therefore, we have

$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2 Z_{i_0}(t_0)+W_{i_0}^q(t_0)\delta^0 Z_{i_0}(t_0)+(q\mu_{i_0}^{q-1}(t_0)\delta^0 V_{i_0}(t_0))Z_{i_0}(t_0)+f(V_{i_0}(t_0),t_0)-f(W_{i_0}(t_0),t_0)\le 0,$$

where $\mu_{i_0}(t_0)\in\left(V_{i_0}(t_0),W_{i_0}(t_0)\right)$, which contradicts the first strict inequality of the lemma and this ends the proof.

Lemma 2.3 Let $U_h(t)$ be the solution of (7)-(10). Then, we have

$$U_i(t)>0 \ \ for \ \ 0\le i\le I,\ t\in\left(0,T_b^h\right). \qquad (22)$$

Proof. Assume that there exists a time $t_0\in\left(0,T_b^h\right)$ such that $U_{i_0}(t_0)=0$ for a certain $i_0\in\{0,\cdots,I\}$.

We observe that

$$\frac{dU_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{U_{i_0}(t_0)-U_{i_0}(t_0-k)}{k}\le 0,\quad 0\le i_0\le I,$$

$$\delta^2 U_{i_0}(t_0)=\frac{U_{i_0+1}(t_0)-2U_{i_0}(t_0)+U_{i_0-1}(t_0)}{h^2}>0,\ 1\le i_0\le I-1,$$

$$\delta^2 U_{i_0}(t_0)=\frac{2U_1(t_0)-2U_0(t_0)}{h^2}>0 \ \ if \ \ i_0=0,$$

$$\delta^2 U_{i_0}(t_0)=\frac{2U_{I-1}(t_0)-2U_I(t_0)}{h^2}>0 \ \ if \ \ i_0=I.$$

which implies that

$$\frac{dU_{i_0}(t_0)}{dt}-\delta^2 U_{i_0}(t_0)+U_{i_0}^q(t_0)\delta^0 U_{i_0}(t_0)-U_{i_0}^p(t_0)<0,\quad 1\le i_0\le I-1,$$

$$\frac{dU_0(t_0)}{dt}-\delta^2 U_0(t_0)-U_0^p(t_0)<0,$$

$$\frac{dU_I(t_0)}{dt}-\delta^2 U_I(t_0)-U_I^p(t_0)<0.$$

But these inequalities contradict (7)-(9) and we obtain the desired result.

Lemma 2.4 Let $U_h(t)$ be the solution of (7)-(10). Then, we have

$$U_{i+1}(t)<U_i(t) \ \ for \ \ 0\le i\le I-1,\ \ t\in\left(0,T_b^h\right). \qquad (23)$$

Proof. Introduce the vector $Z_h(t)$ defined as follows $Z_i(t)=U_{i+1}(t)-U_i(t)$ for $0\le i\le I-1$. Let t_0 be the first $t>0$ such that $Z_i(t)<0$ for $t\in[0,t_0)$ but $Z_{i_0}(t_0)=0$ for a certain $i_0\in\{0,\cdots,I-1\}$. Without loss of generality, we may suppose that i_0 is the smallest integer which satisfies the above equality. It follows that

$$\frac{dZ_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k}\ge 0,\quad 0\le i_0\le I-1,$$

$$\delta^2 Z_{i_0}(t_0)=\frac{Z_{i_0+1}(t_0)-2Z_{i_0}(t_0)+Z_{i_0-1}(t_0)}{h^2}<0,\ 1\le i_0\le I-1,$$

$$\delta^2 Z_{i_0}(t_0)=\frac{2Z_1(t_0)-2Z_0(t_0)}{h^2}<0 \;\; if \;\; i_0=0,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2 Z_{i_0}(t_0)+U^q_{i_0+1}(t_0)\delta^0 Z_{i_0}(t_0)+(q\mu^{q-1}_{i_0}(t_0)\delta^0 U_{i_0}(t_0)-p\beta^{p-1}_{i_0}(t_0))Z_{i_0}(t_0)>0, 1\le i_0\le I-1,$$

$$\frac{dZ_0(t_0)}{dt}-\delta^2 Z_0(t_0)+p\beta_0^{q-1}(t_0)Z_0(t_0)>0.$$

where $\beta_0(t_0)\in(U_1(t_0),U_0(t_0))$ and $\mu_{i_0}(t_0),\beta_{i_0}(t_0)\in(U_{i_0+1}(t_0),U_{i_0}(t_0))$.

Therefore, we have a contradiction because of (7)-(8). This ends the proof.

Lemma 2.5 Let $U_h(t)$ be the solution of (7)-(10). Then, we have

$$\frac{dU_i(t)}{dt}>0 \;\; for \;\; 0\le i\le I, \; t\in\left(0,T_b^h\right).$$

Proof. Consider the vector $Z_h(t)$ with $Z_i(t)=\frac{dU_i(t)}{dt}$, $0\le i\le I$.

Let t_0 be the first $t>0$ such that $Z_i(t)>0$ for $t\in[0,t_0)$ but $Z_{i_0}(t_0)=0$ for a certain $i_0\in\{0,\cdots,I\}$. Without loss of generality, we may suppose that i_0 is the smallest integer which satisfies the above equality. We get

$$\frac{dZ_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k}\le 0, \quad 0\le i_0\le I,$$

$$\delta^2 Z_{i_0}(t_0)=\frac{Z_{i_0+1}(t_0)-2Z_{i_0}(t_0)+Z_{i_0-1}(t_0)}{h^2}>0, \; 1\le i_0\le I-1,$$

$$\delta^2 Z_{i_0}(t_0)=\frac{2Z_1(t_0)-2Z_0(t_0)}{h^2}>0 \;\; if \;\; i_0=0,$$

$$\delta^2 Z_{i_0}(t_0)=\frac{2Z_{I-1}(t_0)-2Z_0(t_0)}{h^2}>0 \;\; if \;\; i_0=I,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2 Z_{i_0}(t_0)+U^q_{i_0}(t_0)\delta^0 Z_{i_0}(t_0)+(qU^{q-1}_{i_0}(t_0)\delta^0 U_{i_0}(t_0)-pU^{p-1}_{i_0}(t_0))Z_{i_0}(t_0)<0 \;\; if \;\; 1\le i_0\le I-1,$$

$$\frac{dZ_0(t_0)}{dt}-\delta^2 Z_0(t_0)-pU_0^{p-1}(t_0)<0,$$

$$\frac{dZ_I(t_0)}{dt}-\delta^2 Z_I(t_0)-pU_I^{p-1}(t_0)<0.$$

But these inequalities contradict (7)-(9) and leads to the desired result.

Lemma 2.6 Let $U_h(t)$ be the solution of (7)-(10). Then, we have, for $p\ge q+1$ such that $q\ge 1$,

$$U_i^{p-1}(t)>-\frac{p(p-1)}{q}hU_{i-1}^{p-2}(t)\delta^0 U_i(t) \;\; for \;\; 1\le i\le I-1, \; t\in\left(0,T_b^h\right).$$

Proof. Define the vectors $Z_h(t),\ K_h(t)\ and\ \ V_h(t)$ such that $Z_i(t)=K_i(t)-V_i(t)$ with $K_i(t)=U_i^{p-1}(t)\ and\ V_i(t)=-\frac{p(p-1)}{q}hU_{i-1}^{p-2}(t)\delta^0U_i(t)\ for\ \ 1\le i\le I-1$. Let t_0 be the first $t>0$ such that $Z_i(t)>0$ for $t\in[0,t_0)$ but $Z_{i_0}(t_0)=0$ for a certain $i_0\in\{1,\cdots,I-1\}$. We may suppose that i_0 is the smallest integer which satisfies the above equality. It follows that

$$\frac{dZ_{i_0}(t_0)}{dt}=\lim_{k\to 0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k}\le 0,\quad 1\le i_0\le I-1,$$

$$\delta^2 Z_{i_0}(t_0)=\frac{Z_{i_0+1}(t_0)-2Z_{i_0}(t_0)+Z_{i_0-1}(t_0)}{h^2}>0,\ 1\le i_0\le I-1,$$

which implies that

$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2 Z_{i_0}(t_0)+K_{i_0}^q(t_0)\delta^0 Z_{i_0}(t_0)+(q\mu_{i_0}^{q-1}(t_0)\delta^0 V_{i_0}(t_0)-p\beta_{i_0}^{p-1}(t_0))Z_{i_0}(t_0)<0,\ 1\le i_0\le I-1,$$

where $\mu_{i_0}(t_0),\beta_{i_0}(t_0)\in\left(V_{i_0}(t_0),K_{i_0}(t_0)\right)$.

But this inequality contradicts (7) and we obtain the desired result.

Lemma 2.7 Let $U_h\in C^1\left([0,T],\Re^{I+1}\right) such\ that U_h>0.$ Then, we have,

$$\delta^2U_i^p\ge pU_i^{p-1}\delta^2U_i\quad for\ \ 0\le i\le I,\ \ p\ge 2.$$

Proof. Using Taylor's expansion, we get

$$\delta^2U_i^p=pU_i^{p-1}\delta^2U_i+(U_{i+1}-U_i)^2\frac{p(p-1)}{2h^2}\xi_i^{p-2}+(U_{i-1}-U_i)^2\frac{p(p-1)}{2h^2}\theta_i^{p-2},\ 1\le i\le I-1,$$

$$\delta^2U_0^p=pU_0^{p-1}\delta^2U_0+(U_1-U_0)^2\frac{p(p-1)}{2h^2}\theta_0^{p-2},$$

$$\delta^2U_I^p=pU_I^{p-1}\delta^2U_I+(U_{I-1}-U_I)^2\frac{p(p-1)}{2h^2}\theta_I^{p-2},$$

Where $p\ge 2,\ \theta_0\in(U_1,U_0),\ \theta_i\in(U_i,U_{i-1}),\ \xi_i\in(U_{i+1},U_i)\ and\ \theta_I\in(U_I,U_{I-1})$.

The result follows taking into account the fact that $U_h>0$.

Lemma 2.8 Let $U_h\in C^1\left([0,T],\Re^{I+1}\right) such\ that U_h>0.$ Then, we have,

$$-U_i\delta^0U_i^p\ge -pU_i^p\delta^0U_i-p(p-1)hU_{i-1}^{p-2}(\delta^0U_i)^2,\ 1\le i\le I-1,\ \ p\ge 2.$$

Proof. Applying Taylor's expansion, we obtain

$$\delta^0U_i^p=pU_{i-1}^{p-1}\delta^0U_i+(U_{i+1}-U_{i-1})^2\frac{p(p-1)}{4h}U_{i-1}^{p-2},\ 1\le i\le I-1,\ \ p=2,$$

$$\delta^0U_i^p=pU_{i-1}^{p-1}\delta^0U_i+(U_{i+1}-U_{i-1})^2\frac{p(p-1)}{4h}U_{i-1}^{p-2}+(U_{i+1}-U_{i-1})^3\frac{p(p-1)(p-2)}{12h}\zeta_i^{p-3},1\le i\le I-1,\ p\ge 3,$$

Where $\zeta_i\in(U_{i+1},U_{i-1})$.

Using Lemma 2.4 and $U_h>0$, we have the desired result.

3. Semidiscrete Blow-up solutions

In this section under some assumptions, we show that the solution U_h of (7)-(10) blows up in a finite time and estimate its semidiscrete blow-up time.

Theorem 3.1 Let U_h be the solution of (7)-(10), then the solution U_h blows up in a finite time T_b^h with following estimate

$$T_b^h \le \frac{1}{(p-1)} \frac{1}{(\min_{0\le i\le I}(\varphi_i))^{p-1}}. \tag{24}$$

Proof. Consider the following differential equation

$$\dot{\alpha}(t) = \alpha^p(t), t \in (0, T_\alpha), \ p \ge 2, \tag{25}$$

$$\alpha(0) = \min_{0\le i\le I}(\varphi_i), \tag{26}$$

with $T_\alpha = \frac{1}{(p-1)} \frac{1}{(\min_{0\le i\le I}(\varphi_i))^{p-1}}$.

Introduce the vector $V_h(t)$ such that $V_i(t) = \alpha(t), 0 \le i \le I, \ t \in (0, T_\alpha)$. Let the vector $Z_h(t)$ define as follow

$Z_h(t) = U_h(t) - V_h(t)$. It not hard to see that

$$\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) + U_i^q(t)\delta^0 Z_i(t) + (q\mu_i^{q-1}(t)\delta^0 V_i(t) - p\beta_i^{p-1}(t))Z_i(t) \ge 0 \ if \quad 0 \le t \le I, t \in (0, T_1),$$

$$Z_i(0) = 0,$$

where $\mu_i(t), \beta_i(t) \in (V_i(t), U_i(t))$ and $T_1 = \min\{T_\alpha, T_b^h\}$.

Due to Lemma 2.2, we have $U_i(t) \ge V_i(t), \ 0 \le t \le I, \ t \in (0, T_1)$. We deduce that

$$T_b^h \le T_\alpha \le \frac{1}{(p-1)} \frac{1}{(\min_{0\le i\le I}(\varphi_i))^{p-1}}.$$

The following theorem gives a best result than the previous.

Theorem 3.2 Let U_h be the solution of (7)-(10). Suppose that there exists a positive integer λ such that

$$\delta^2 U_i(0) - U_i^q(0)\delta^0 U_i(0) + U_i^p(0) \ge \lambda U_i^p(0), \quad 0 \le i \le I. \tag{27}$$

Then, the solution U_h blows up in a finite time T_b^h and we have the following estimate

$$T_b^h \le \frac{1}{\lambda} \frac{\|U_h(0)\|_\infty^{1-p}}{(p-1)}.$$

Proof. Let $(0, T_b^h)$ be the maximal time interval on which $\|U_h(t)\|_\infty < \infty$. Our aim is to show that T_b^h is finite and satisfies the above inequality. Introduce the vector $J_h(t)$ such that

$$J_i(t) = \frac{dU_i(t)}{dt} - \lambda U_i^p(t), \quad 0 \le i \le I. \tag{28}$$

A straightforward calculation gives

$$\frac{dJ_i}{dt}-\delta^2 J_i=\frac{d^2U_i}{dt^2}-\lambda pU_i^{p-1}\frac{dU_i}{dt}-\delta^2\frac{dU_i}{dt}+\lambda\delta^2U_i^p, 1\le i\le I-1.$$

From Lemma 2.7, we have $\delta^2U_i^p\ge pU_i^{p-1}\delta^2U_i$ *for* $0\le i\le I,\ p\ge 2$, which implies that

$$\frac{dJ_i}{dt}-\delta^2 J_i\ge\frac{d}{dt}(\frac{d}{dt}U_i-\delta^2U_i)-\lambda pU_i^{p-1}(\frac{d}{dt}U_i-\delta^2U_i), 1\le i\le I-1,$$

$$\frac{dJ_i}{dt}-\delta^2 J_i\ge\frac{d}{dt}(-U_i^q\delta^0U_i+U_i^p)-\lambda pU_i^{p-1}(-U_i^q\delta^0U_i+U_i^p),\ 1\le i\le I-1.$$

Using (7)-(9), we arrive at

$$\frac{dJ_i}{dt}-\delta^2 J_i+U_i^q\delta^0J_i+(qU_i^{q-1}\delta^0U_i-pU_i^{p-1})J_i\ge\lambda(p-q)U_i^{q-1}U_i^p\delta^0U_i-\lambda U_i^q\delta^0U_i^p, 1\le i\le I-1.$$

$$\frac{dJ_0}{dt}-\delta^2 J_0-pU_0^{p-1}J_0\ge 0,$$

$$\frac{dJ_I}{dt}-\delta^2 J_I-pU_I^{p-1}J_I\ge 0.$$

Using Lemma 2.8 and the fact that $p > q$, we get

$$\frac{dJ_i}{dt}-\delta^2 J_i+U_i^q\delta^0J_i+(qU_i^{q-1}\delta^0U_i-pU_i^{p-1})J_i\ge-\lambda U_i^q\delta^0U_i(qU_i^{p-1}+p(p-1)hU_{i-1}^{p-2}\delta^0U_i), 1\le i\le I-1,$$

$$\frac{dJ_0}{dt}-\delta^2 J_0-pU_0^{p-1}J_0\ge 0,$$

$$\frac{dJ_I}{dt}-\delta^2 J_I-pU_I^{p-1}J_I\ge 0.$$

From Lemma 2.6, we have $U_i^{p-1}>-\frac{p(p-1)}{q}hU_{i-1}^{p-2}\delta^0U_i$ *for* $1\le i\le I-1$ and thanks to $-\lambda U_i^q\delta^0U_i\ge 0$, we get finally

$$\frac{dJ_i}{dt}-\delta^2 J_i+U_i^q\delta^0J_i+(qU_i^{q-1}\delta^0U_i-pU_i^{p-1})J_i\ge 0,\ 1\le i\le I-1,$$

$$\frac{dJ_0}{dt}-\frac{(2J_1-2J_0)}{h^2}-pU_0^{p-1}J_0\ge 0,$$

$$\frac{dJ_I}{dt}-\frac{(2J_{I-1}-2J_I)}{h^2}-pU_I^{p-1}J_I\ge 0.$$

From (27), we observe that

$$J_i(0)=\delta^2U_i(0)-U_i^q(0)\delta^0U_i(0)+U_i^p(0)-\lambda U_i^p(0)\ge 0,\quad 0\le i\le I.$$

We deduce from Lemma 2.1 that $J_h(t)\ge 0$ *for* $t\in(0,T_b^h)$, which implies that

$$\frac{dU_i(t)}{dt}\ge\lambda U_i^p(t),\quad 0\le i\le I,\ t\in(0,T_b^h). \tag{29}$$

These estimates may be rewritten in the following form

$$U_i^{-p}dU_i\ge\lambda dt,\quad 0\le i\le I.$$

Integrating the above inequalities over (t,T_b^h), we arrive at

$$T_b^h-t\le\frac{1}{\lambda}\frac{(U_i(t))^{1-p}}{(p-1)}. \tag{30}$$

which implies that

$$T_b^h \le \frac{1}{\lambda}\frac{\|U_h(0)\|_\infty^{1-p}}{(p-1)}.$$

Remark 3.1 The inequality (30) implies that

$$T_b^h - t_0 \le \frac{1}{\lambda}\frac{\|U_h(t_0)\|_\infty^{1-p}}{(p-1)} \quad if \quad 0 < t_0 < T_b^h.$$

4. Convergence of the semidiscrete blow-up time

In this section, under some assumptions, we show that the semidiscrete blow-up time converges to the real one when the mesh size goes to zero. In order to obtain the convergence of semidiscrete blow-up time, we firstly prove the following theorem about the convergence of the semidiscrete scheme.

Theorem 4.1 Assume that (1)-(3) has a solution $u \in C^{4,1}([0,1]\times[0,T])$ and the initial condition at (10) satisfies

$$\left\|U_h^0 - u_h(0)\right\|_\infty = o(1) \text{ as } h \to 0, \tag{31}$$

Where $u_h(t) = (u(x_0,t),\cdots,u(x_I,t))^T$. Then, for h sufficiently small, the problem (7)-(10) has a unique solution $U_h \in C^1([0,T],\Re^{I+1})$ such that

$$\max_{0\le t\le T}\|U_h(t)-u_h(t)\|_\infty = O(\left\|U_h^0 - u_h(0)\right\|_\infty + h^2) \text{ as } h \to 0. \tag{32}$$

Proof. Let $K > 0$ be such that

$$\|u\|_\infty \le K. \tag{33}$$

The problem (7)-(10) has for each h, a unique solution $U_h \in C^1([0,T_b^h],\Re^{I+1})$. Let *t(h)* the greatest
value of $t > 0$ such that

$$\|U_h(t)-u_h(t)\|_\infty < 1 \text{ for } t \in (0,t(h)). \tag{34}$$

The relation (31) implies that $t(h) > 0$ for h sufficiently small. Let $t^*(h) = \min\{t(h),T\}$. By the triangular inequality, we obtain

$$\|U_h(t)\|_\infty \le \|u(.,t)\|_\infty + \|U_h(t)-u_h(t)\|_\infty \text{ for } t \in (0,t^*(h)),$$

which implies that

$$\|U_h(t)\|_\infty \le 1+K \text{ for } t \in (0,t^*(h)). \tag{35}$$

Let $e_h(t) = U_h(t) - u_h(t)$ be the error of discretization. Using Taylor's expansion, we have for $t \in (0,t^*(h))$,

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) + u^q(x_i,t)\delta^0 e_i(t) = (p\beta_i^{p-1}(t) - q\mu_i^{q-1}(t)\delta^0 u(x_i,t))e_i(t) - \frac{h^2}{6}u^q(x_i,t)u_{xxx}(\tilde{x}_i,t),$$

$$\frac{de_0(t)}{dt} - \frac{(2e_1(t)-2e_0(t))}{h^2} = p\beta_0^{p-1}(t)e_0(t) + \frac{h^2}{12}u_{xxxx}(\tilde{x}_0,t),$$

$$\frac{de_I(t)}{dt}-\frac{(2e_{I-1}(t)-2e_I(t))}{h^2}=p\beta_I^{p-1}(t)e_I(t)+\frac{h^2}{12}u_{xxxx}(\tilde{x}_I,t),$$

where $\mu_i(t)$ and $\beta_i(t)$ are intermediate values between $u(x_i,t)$ and $U_i(t)$ for $i\in\{0,...,I\}$.

Using (35), there exists a constant $M>0$ such that

$$\frac{de_i(t)}{dt}-\delta^2 e_i(t)+u^q(x_i,t)\delta^0 e_i(t)\le M|e_i(t)|+Mh^2,\ 1\le i\le I-1, \tag{36}$$

$$\frac{de_0(t)}{dt}-\frac{(2e_1(t)-2e_0(t))}{h^2}\le M|e_0(t)|+Mh^2, \tag{37}$$

$$\frac{de_I(t)}{dt}-\frac{(2e_{I-1}(t)-2e_I(t))}{h^2}\le M|e_I(t)|+Mh^2. \tag{38}$$

Consider the vector W_h such that

$$W_i(t)=e^{(M+1)t}(\|U_h^0-u_h(0)\|_\infty+Mh^2)\ ,\ 0\le i\le I\ .$$

A direct calculation yields

$$\frac{dW_i(t)}{dt}-\delta^2 W_i(t)+u^q(x_i,t)\delta^0 W_i(t)>M|W_i(t)|+Mh^2,\ 1\le i\le I-1, \tag{39}$$

$$\frac{dW_0(t)}{dt}-\frac{(2W_1(t)-2W_0(t))}{h^2}>M|W_0(t)|+Mh^2, \tag{40}$$

$$\frac{dW_I(t)}{dt}-\frac{(2W_{I-1}(t)-2W_I(t))}{h^2}>M|W_I(t)|+Mh^2, \tag{41}$$

$$W_i(0)>e_i(0)\ ,\ 0\le i\le I\ . \tag{42}$$

It follows from Lemma 2.2 that

$W_i(t)>e_i(t)$ for $t\in(0,t^*(h)),\ 0\le i\le I$.

By the same way, we also prove that

$W_i(t)>-e_i(t)$ for $t\in(0,t^*(h)),\ 0\le i\le I$,

which implies that

$W_i(t)>|e_i(t)|$ for $t\in(0,t^*(h)),\ 0\le i\le I$.

We deduce that

$$\|U_h(t)-u_h(t)\|_\infty\le e^{(M+1)T}(\|U_h^0-u_h(0)\|_\infty+Mh^2)\ ,\ t\in(0,t^*(h)).$$

Let us show that $t^*(h)=T$. Suppose that $T>t^*(h)$. From (34), we obtain

$$1=\|U_h(t(h))-u_h(t(h))\|_\infty\le e^{(M+1)T}(\|U_h^0-u_h(0)\|_\infty+Mh^2)\ . \tag{43}$$

Since $e^{(M+1)T}(\|U_h^0-u_h(0)\|_\infty+Mh^2)\to 0\ when\, h\to 0,$ we deduce from (43) that 1≤0, which is impossible. Consequently $t^*(h)=T$, and we conclude the proof.

Theorem 4.2 Suppose that the solution u of (1)-(3) blows up in a finite time T_b such that $u\in C^{4,1}([0,1]\times[0,T_b])$ and the initial condition at (10) satisfies

$$\|U_h^0-u_h(0)\|_\infty=o(1)\ \text{as}\ h\to 0\ . \tag{44}$$

Assume that there exists a constant λ >0 such that

$$\delta^2 U_i(0) - U_i^q(0)\delta^0 U_i(0) + U_i^p(0) \geq \lambda U_i^p(0), \quad 0 \leq i \leq I. \tag{45}$$

Then the solution U_h of (7)-(10) blows up in a finite time T_b^h and

$$\lim_{h\to 0} T_b^h = T_b. \tag{46}$$

Proof. Let $\varepsilon > 0$. There exists N such that

$$\frac{1}{\lambda}\frac{y^{1-p}}{(p-1)} \leq \frac{\varepsilon}{2} < \infty \ \ for \quad y \in [N, +\infty[. \tag{47}$$

Since $\lim_{t\to T_b} \max_{x\in[0,1]} |u(x,t)| = +\infty$, then, there exists T_1 such that

$|T_1 - T_b| \leq \frac{\varepsilon}{2}$ *and* $\|u(x,t)\|_\infty \geq 2N$ *for* $t \in [T_1, T_b]$. Let $T_2 = \frac{T_1 + T_b}{2}$, then $\sup_{t\in[0,T_2]} |u(x,t)| < +\infty$.

It follows from Theorem 4.1 that $\sup_{t\in[0,T_2]} |U_h(t) - u_h(t)| < N$. Applying the triangular inequality, we get

$\|U_h(t)\|_\infty \geq \|u(.,t)\|_\infty - \|U_h(t) - u_h(t)\|_\infty$, which leads to $\|U_h(t)\|_\infty \geq N$ for $t \in [0, T_2]$.

From Theorem 3.2, $U_h(t)$ blows up at the time T_b^h. We deduce from Remark 3.1 and (47) that

$$|T_b - T_b^h| \leq |T_b - T_2| + |T_b^h - T_2| \leq \frac{\varepsilon}{2} + \frac{1}{\lambda}\frac{\|U_h(T_2)\|_\infty^{1-p}}{(p-1)} \leq \varepsilon,$$

which leads us to the desired result.

5. Numerical results

In this section, we present some numerical approximations to the blow-up time of (1)-(3). We use the following explicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n)} - 2U_i^{(n)} + U_{i-1}^{(n)}}{h^2} - (U_i^{(n)})^q \frac{U_{i+1}^{(n)} - U_{i-1}^{(n)}}{2h} + (U_i^{(n)})^p, \ 1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n)} - 2U_0^{(n)}}{h^2} + (U_0^{(n)})^p,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n)} - 2U_I^{(n)}}{h^2} + (U_I^{(n)})^p,$$

where $n \geq 0$, $q \geq 1$, $p \geq q+1$, $\Delta t_n = \min\left\{\frac{h^2}{2}, \tau \|U_h^{(n)}\|_\infty^{1-p}\right\}$ *with* $\tau = cont \in (0,1)$.

Also we use the implicit scheme

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} = \frac{U_{i+1}^{(n+1)} - 2U_i^{(n+1)} + U_{i-1}^{(n+1)}}{h^2} - (U_i^{(n)})^q \frac{U_{i+1}^{(n+1)} - U_{i-1}^{(n+1)}}{2h} + (U_i^{(n)})^p, \ 1 \leq i \leq I-1,$$

$$\frac{U_0^{(n+1)} - U_0^{(n)}}{\Delta t_n} = \frac{2U_1^{(n+1)} - 2U_0^{(n+1)}}{h^2} + (U_0^{(n)})^p,$$

$$\frac{U_I^{(n+1)} - U_I^{(n)}}{\Delta t_n} = \frac{2U_{I-1}^{(n+1)} - 2U_I^{(n+1)}}{h^2} + (U_I^{(n)})^p,$$

where $n \geq 0$, $q \geq 1$, $p \geq q+1$, $\Delta t_n = \tau \left\| U_h^{(n)} \right\|_\infty^{1-p}$ *with* $\tau = cont \in (0,1)$.

In the tables 1-18, in rows, we present the numerical blow-up times, numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512, 1024. The numerical blow-up time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ is computed at the first time when $\Delta t_n = \left| T^{n+1} - T^n \right| \leq 10^{-16}$. The order(s) of the method is computed from $s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}$. .

First case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p}$, $q = 1$, $p = 4$ *and* $\tau = \frac{h^2}{2}$.

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	0.041830	4872	-	-
32	0.041707	18527	-	-
64	0.041677	70305	-	2.00
128	0.041669	266066	4	2.00
256	0.041667	1003680	30	2.00
512	0.041667	3772429	223	2.00
1024	0.041667	14120612	1659	2.00

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	0.041830	4872	-	-
32	0.041707	18527	1	2.00
64	0.041677	70305	1	2.00
128	0.041669	266066	8	2.00
256	0.041667	1003680	57	2.00
512	0.041667	3772429	422	2.00
1024	0.041667	14120612	3185	2.00

Second case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p}$, $q = 2$, $p = 4$ *and* $\tau = \frac{h^2}{2}$.

Table 3: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	0.041830	4872	-	-
32	0.041707	18527	-	-
64	0.041677	70305	-	2.00
128	0.041669	266066	4	2.00
256	0.041667	1003680	30	2.00
512	0.041667	3772429	225	2.00
1024	0.041667	14120612	1677	2.00

Table 4: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	0.041830	4872	-	-
32	0.041707	18527	1	-
64	0.041677	70305	1	2.00
128	0.041669	266066	7	2.00
256	0.041667	1003680	56	2.00
512	0.041667	3772429	424	2.00
1024	0.041667	14120612	3195	2.00

Third case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p}, \ q = 3, \ p = 4 \ and \ \tau = \frac{h^2}{2}.$

Table 5: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	0.041830	4872	-	-
32	0.041707	18527	-	-
64	0.041677	70305	-	2.00
128	0.041669	266066	4	2.00
256	0.041667	1003680	30	2.00
512	0.041667	3772429	225	2.00
1024	0.041667	14120612	1677	2.00

Table 6: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	0.041830	4872	-	-
32	0.041707	18527	1	-
64	0.041677	70305	1	2.00
128	0.041669	266066	7	2.00
256	0.041667	1003680	56	2.00
512	0.041667	3772429	424	2.00
1024	0.041667	14120612	3195	2.00

Fourth case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{6-p} + \left(1-(ih)^2\right)^2,\ q=1,\ p=4\ and\ \tau = \frac{h^2}{2}.$

Table 7: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	0.437081	5432	-	-
32	0.435522	20774	-	-
64	0.435133	79339	-	2.00
128	0.435133	302354	5	2.00
256	0.435011	1149416	35	2.00
512	0.435005	4357571	260	2.00
1024	0.435003	16469316	1966	2.00

Table 8: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	0.436323	5433	-	-
32	0.435333	20776	1	-
64	0.435085	79340	2	2.00
128	0.435023	302355	9	2.00
256	0.435008	1149417	64	2.00
512	0.435004	4357572	479	2.00
1024	0.435003	16469317	3649	2.00

Fifth case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{6-p} + \left(1-(ih)^2\right)^2,\ q=2,\ p=4\ and\ \tau = \frac{h^2}{2}.$

Table 9: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	0.434179	5380	-	-
32	0.432479	20774	-	-
64	0.432054	78718	-	1.99
128	0.431948	299088	4	2.00
256	0.431921	1138980	34	2.00
512	0.431915	4313040	255	2.00
1024	0.431913	16283086	1919	2.00

Table 10: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	0.433243	5381	-	-
32	0.432245	20675	-	-
64	0.431996	78718	1	2.00
128	0.431933	299088	8	2.00
256	0.431918	1138980	63	2.00
512	0.431914	4313041	478	2.00
1024	0.431913	16283087	3646	2.00

Sixth case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{6-p} + \left(1-(ih)^2\right)^2,\ q=3,\ p=4\ and\ \tau = \frac{h^2}{2}.$

Table 11: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	0.424625	6578	-	-
32	0.423296	20523	-	-
64	0.422849	83307	-	1.70
128	0.422733	298051	4	1.93
256	0.422703	1134998	34	1.99
512	0.422696	4297701	257	2.00
1024	0.422694	16221699	1930	2.00

Table 12: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	0.423016	6269	-	-
32	0.423016	21837	-	-
64	0.422849	83307	-	1.60
128	0.422715	298183	9	1.88
256	0.422695	1134998	34	1.99
512	0.422696	4297701	477	2.00
1024	0.422694	16221699	3638	2.00

Fourth case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{30-p} + \left(1-(ih)^2\right)^2,\ q = 1,\ p = 4\ and\ \tau = \frac{h^2}{2}.$

Table 13: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	1.504081	5809	-	-
32	1.500052	22268	-	-
64	1.499043	85263	1	1.99
128	1.498791	325888	5	1.99
256	1.498728	1242964	37	1.99
512	1.498712	4729565	280	2.00
1024	1.498708	17949146	2109	2.00

Table 14: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	1.502383	5507	-	-
32	1.499623	21061	-	-
64	1.498936	80435	1	2.00
128	1.498764	306580	8	2.00
256	1.498721	1165728	64	2.00
512	1.498710	4420616	484	2.00
1024	1.498708	16713353	3697	2.00

Eighth case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{30-p} + \left(1-(ih)^2\right)^2,\ q = 2,\ p = 4\ and\ \tau = \frac{h^2}{2}.$

Table 15: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	1.687438	5902	-	-
32	1.684238	22642	-	-
64	1.683437	86766	1	1.99
128	1.683237	331932	5	1.99
256	1.683187	1267138	38	1.99
512	1.683174	4826259	285	2.00
1024	1.683171	18335922	2155	2.00

Table 16: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	1.685276	5526	-	-
32	1.683688	21140	-	-
64	1.683299	80759	1	2.00
128	1.683202	307907	9	2.00
256	1.683178	1171037	64	2.00
512	1.683172	4441851	487	2.00
1024	1.683170	16798297	3716	2.00

Ninth case: $U_i^{(0)} = \left(\frac{1}{2}\right)^{30-p} + \left(1-(ih)^2\right)^2,\ q=3,\ p=4\ and\ \tau = \frac{h^2}{2}.$

Table 17: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	1.808074	5951	-	-
32	1.805591	22934	-	-
64	1.804970	87766	1	1.99
128	1.804815	335916	5	1.99
256	1.804776	1283075	39	1.99
512	1.804766	4890007	293	2.00
1024	1.804764	18590918	2206	2.00

Table 18: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	CPUtime	s
16	1.805737	5452	-	-
32	1.804992	21109	-	-
64	1.804819	80949	1	2.10
128	1.804777	308684	9	2.02
256	1.804766	1174144	64	2.00
512	1.804764	4454280	489	2.00
1024	1.804763	16848009	4842	2.00

In the following, we also give some plots to illustrate our analysis. For the different plots, we used both explicit and implicit schemes in the case where I=16, q=3 and p=4. In Figures 1 and 2 we can appreciate that the discrete solution blows up globally in a finite time where the initial data is a constant. In Figures 3, 4, 5 and 6, we see that the blow-up is local when the initial data is not a constant. The Figures 7, 8, 9, 10, 11and 12 show the effect of the convection term on the evolution of the solution. In Figures 13, 14, 15, 16, 17 and 18, we observe that the solution of our problem blows up in a finite time, when the initial data is $\left(\frac{1}{2}\right)^{3-p}$, $\left(\frac{1}{2}\right)^{6-p}+\left(1-(ih)^2\right)^2$ or $\left(\frac{1}{2}\right)^{30-p}+\left(1-(ih)^2\right)^2$ with $i\in\{0,\cdots,I\}$

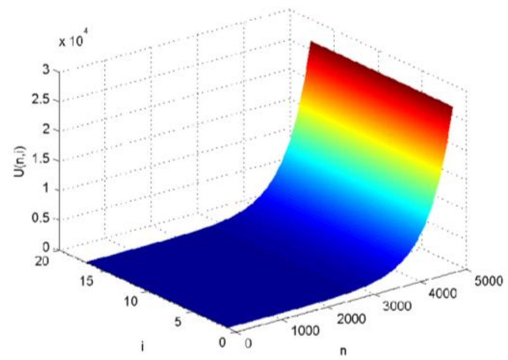

Figure 1: Evolution of the discrete solution (Explicit scheme) $U_i^{(0)}=\left(\frac{1}{2}\right)^{3-p}$, $q=3,\ p=4.$

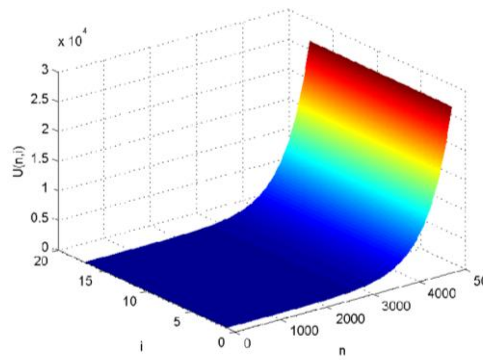

Figure 2: Evolution of the discrete solution (Implicit scheme) $U_i^{(0)}=\left(\frac{1}{2}\right)^{3-p}$, $q=3,\ p=4.$

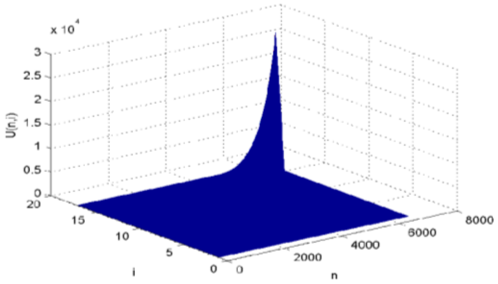

Figure 3: Evolution of the discrete solution (Explicit scheme) $U_i^{(0)} = \left(\frac{1}{2}\right)^{6-p} + \left(1-(ih)^2\right)^2,\ q = 3,\ p = 4.$

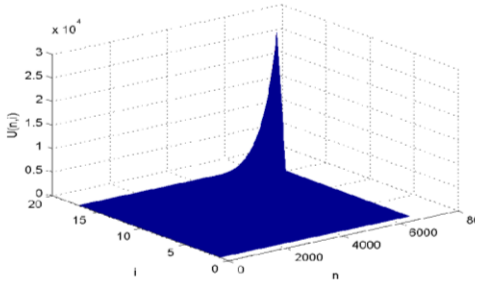

Figure 4: Evolution of the discrete solution (Implicit scheme) $U_i^{(0)} = \left(\frac{1}{2}\right)^{6-p} + \left(1-(ih)^2\right)^2,\ q = 3,\ p =$

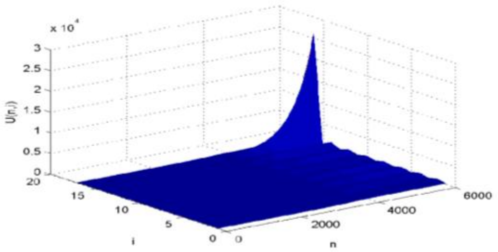

Figure 5: Evolution of the discrete solution (Explicit scheme) $U_i^{(0)} = \left(\frac{1}{2}\right)^{30-p} + \left(1-(ih)^2\right)^2,\ q = 3,\ p = 4.$

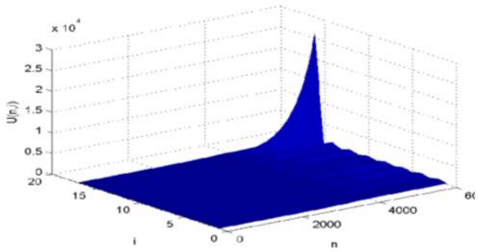

Figure 6: Evolution of the discrete solution (Implicit scheme) $U_i^{(0)} = \left(\frac{1}{2}\right)^{30-p} + \left(1-(ih)^2\right)^2,\ q = 3,\ p =$

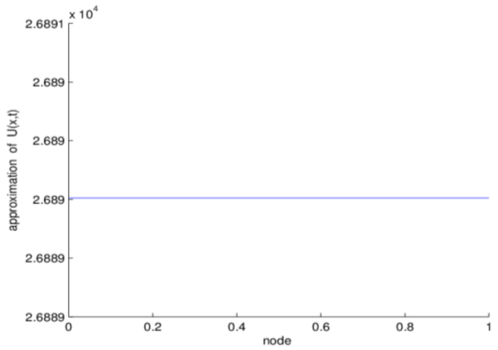

Figure 7: Evolution of U(x,t) according to the node (explicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p},\ q = 3,\ p = 4.$

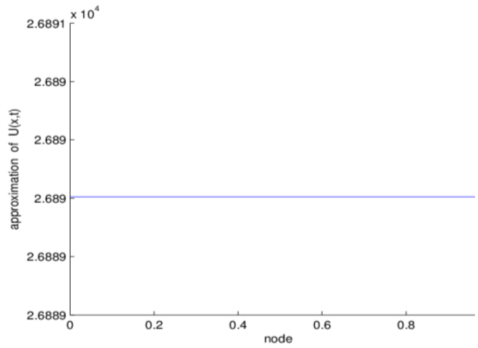

Figure 8: Evolution of U(x,t) according to the node (implicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p},\ q = 3,\ p = 4.$

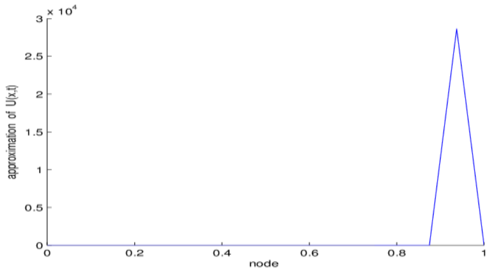

Figure 9: Evolution of U(x,t) according to the node (explicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{6-p} + \left(1-(ih)^2\right)^2,\ q=3,\ p=4.$

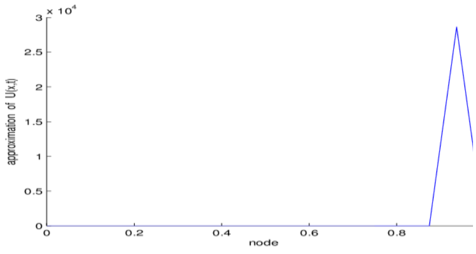

Figure 10: Evolution of U(x,t) according to the node (implicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{6-p} + \left(1-(ih)^2\right)^2,\ q=3,\ p=4$

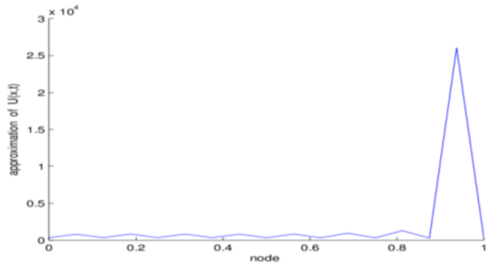

Figure 11: Evolution of U(x,t) according to the node (explicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{30-p} + \left(1-(ih)^2\right)^2,\ q=3,\ p=4.$

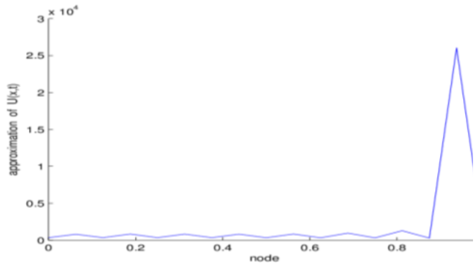

Figure 12: Evolution of U(x,t) according to the node (implicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{30-p} + \left(1-(ih)^2\right)^2,\ q=3,\ p=4$

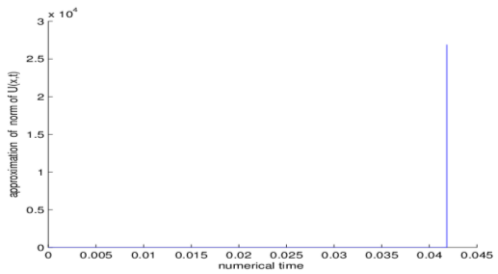

Figure 13: Evolution of U(x,t) according to the time (explicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p},\ q=3,\ p=4.$

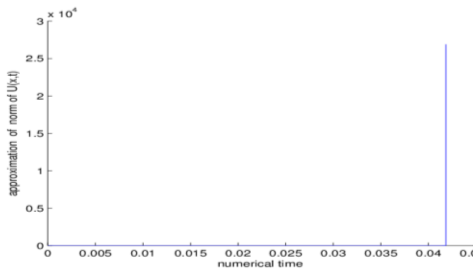

Figure 14: Evolution of U(x,t) according to the time (implicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{3-p},\ q=3,\ p=4.$

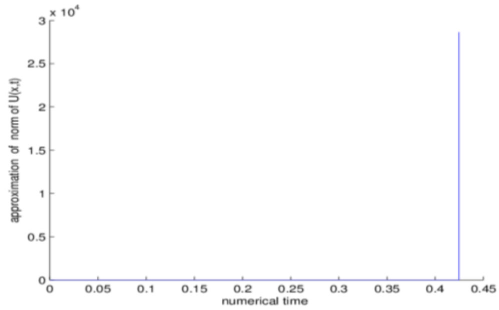

Figure 15: Evolution of U(x,t) according to the time (explicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{6-p} + \left(1-(ih)^2\right)^2$, $q = 3, p = 4.$

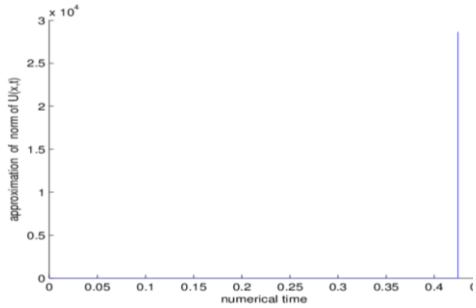

Figure 16: Evolution of U(x,t) according to the time (implicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{6-p} + \left(1-(ih)^2\right)^2$, $q = 3, p = 4.$

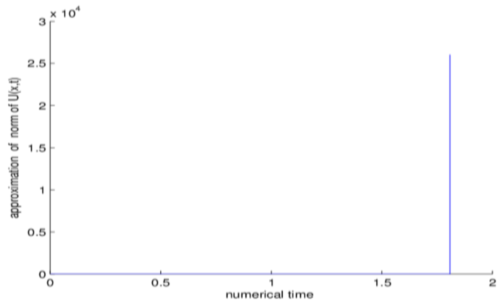

Figure 17: Evolution of U(x,t) according to the time (explicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{30-p} + \left(1-(ih)^2\right)^2$, $q = 3, p = 4.$

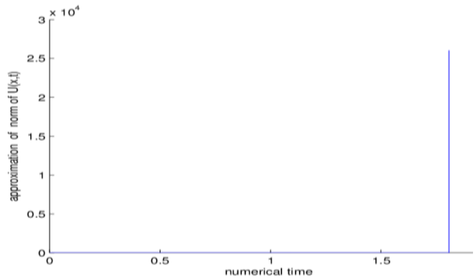

Figure 18: Evolution of U(x,t) according to the time (implicit scheme), $U_i^{(0)} = \left(\frac{1}{2}\right)^{30-p} + \left(1-(ih)^2\right)^2$, $q = 3, p = 4$

Remark 5.1

We observe that, the solution of our problem blows up in a finite time for all $p \geq q+1$ such that $q \geq 1$.

***First case*:** the initial data is the constant $\left(\frac{1}{2}\right)^{3-p}$.

When q approaches $p\,(q < p)$, the blow-up is global. In this case, the convection term, which is null, has no turbulence effect on the blow-up created by the reaction term. (*No turbulent blow-up*).

Second case: the initial data is $\left(\frac{1}{2}\right)^{6-p} + \left(1-(ih)^2\right)^2$.

When q approaches $p\,(q < p)$, the blow-up is local with a blow-up time more and more small. The convection term accelerates the blow-up created by the reaction term. (*No turbulent blow-up*).

***Third case*:** the initial data is $\left(\frac{1}{2}\right)^{30-p} + \left(1-(ih)^2\right)^2$.

When q approaches $p\,(q < p)$, the blow-up is local with a blow-up time more and more greater. The convection term, responsible of the turbulence, delays the blow-up created by the reaction term. (*Turbulent blow-up*).

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