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Properties Of Prime Ternary Γ-Radicals In Ternary Γ-Semigroups

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ABSTRACT:

This paper is divided into 3 sections. In section 1, the terms, 'completely prime ternary Γ-ideal' and 'prime ternary Γ-ideal' in a ternary Γ-semigroup are introduced. It is proved that in a ternary Γ-semigroup (i) A is a prime ternary Γ-ideal of T, (ii) For *a*, *b*, *c* ∈ T; <*a*>Γ<*b*>Γ<*c*> ⊆ A implies *a* ∈ A or *b* ∈ A or *c* ∈ A, (iii) For *a*; *b*; *c* ∈ T; $T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1\Gamma b\Gamma T^1\Gamma T^1\Gamma c\Gamma T^1\Gamma T^1 \subseteq A$ implies *a* ∈ A or *b* ∈ A or *c* ∈ A are equivalent. It is proved that a ternary Γ-ideal A of a ternary Γ-semigroup T is (1) completely prime iff T\P is either a ternary Γ-subsemigroup of T or empty (2) prime iff T\P is either an *m*-system or empty. It is also proved that every completely prime ternary Γ-ideal of a ternary Γ-semigroup is prime. In a globally idempotent ternary Γ-semigroup, it is proved that every maximal ideal is prime. It is also proved that a globally idempotent ternary Γ-semigroup having a maximal Ternary Γ-ideal contains semisimple elements.

In section 2, the terms, completely semiprime ternary Γ-ideal and semiprime ternary Γ-ideal in a ternary Γ-semigroup are introduced and characterized them.

In section 3, to each ternary Γ-ideal A of a ternary Γ-semigroup T, we associate four types of sets namely A_1, A_2, A_3, A_4 and we proved that $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$. In a commutative ternary Γ-semigroup, it is proved that $A_1 = A_2 = A_3 = A_4$ and in general ternary Γ-semigroup, it is proved that $A_1 \neq A_2 \neq A_3 \neq A_4$ by means of examples. The terms 'Γ-radical' and 'complete Γ-radical' of a ternary Γ-ideal in a ternary Γ-semigroup are also introduced and some of their properties are obtained. Iit is proved that if A is a semiprime ternary Γ-ideal of a ternary Γ-semigroup T and M is a maximal *m*-system of T such that $A \cap M = \varnothing$, then T\M is a minimal prime ternary Γ-ideal of T containing A.

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INTRODUCTION:

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, and the like. The theory of ternary algebraic systems was introduced by LEHMER in 1932, but earlier such structures was studied by KASNER who give the idea of n-ary algebras. LEHMER investigated certain algebraic systems called triplexes which turn out to be commutative ternary groups. Ternary semigroups are universal algebras with one associative ternary operation. The notion of ternary semigroup was known to BANACH who is credited with example of a ternary semigroup which can not reduce to a semigroup. SIOSON introduced the ideal theory in ternary semigroups. SANTIAGO developed the theory of ternary semigroups. He studied regular and completely regular ternary semigroups. M. SHABIR and A. KHAN studied prime ideals and prime one sided ideals in semigroup. SHABIR and BASHIR launched prime ideals in ternary semigroups. A. Anjaneyulu [1] introduced the study of pseudo symmetric ideals in semigroups D. Madhusudhana Rao and A. Anjaneyulu [2], [3] studied about Γ-semigroups. Further D. Madhusudhana Rao and A. Anjaneyulu and Y. Sarla [10] extended the same results to ternary semigroups. Madhusudhana Rao and Srinivasa Rao [5, [6], [7] studied about ternary semirings. In this paper mainly we extended the same results to ternary Γ-semigroups.

2. PRELIMINARIES

Definition 2. 1: Let T and Γ be two non-empty set. Then T is said to be a ***Ternary* Γ-*semigroup*** if there exist a mapping from T×Γ×T×Γ×T to T which maps $(x_1, \alpha, x_2, \beta, x_3) \rightarrow x_1\alpha x_2\beta x_3$ satisfying the condition $[x_1\alpha x_2\beta x_3 \; \gamma x_4\delta x_5] = [x_1\alpha \; x_2\beta x_3\gamma x_4 \; \delta x_5] = [x_1\alpha x_2\beta \; x_3\gamma x_4\delta x_5] \; \forall \; x_i \in$ T, $1 \le i \le 5$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Note 2. 2: For the convenience we write $x_1\alpha x_2\beta x_3$ instead of $x_1\alpha x_2\beta x_3$

Note 2. 3: Let T be a ternary Γ-semigroup. If A, B and C are three subsets of T, we shall denote the set AΓBΓC = $a\alpha b\beta c : a \in A, b \in B, c \in C, \alpha, \beta \in \Gamma$.

Note 2. 4: Any Γ-semigroup can be reduced to a ternary Γ-semigroup.

Example: **2. 5:** Let T = { 5*n* + 4: *n* is a positive integer} and Γ = { 5*n* + 1: n is a positive integer}. Then T is a ternary Γ-semigroup with the operation defined by $a\alpha b\beta c = a + \alpha + b + \beta + c$ where $a, b, c \in$ S, $\alpha, \beta \in \Gamma$ and + is the usual addition of integers.

Definition 2. 6: An element *a* of a ternary Γ-semigroup T is said to be an ***identity*** provided $a\alpha a\beta t = t\alpha a\beta a = a\alpha t\beta a = t \; \forall \; t \in$ T, $\alpha, \beta \in \Gamma$.

Notation 2. 7: Let T be a ternary Γ-semigroup. If T has an identity, let T^1 = T and if T does not have an identity, let T^1 be the ternary semigroup T with an identity adjoined usually denoted by the symbol 1.

Definition 2. 8: A nonempty subset A of a ternary Γ-semigroup T is said to be ***ternary Γ-ideal*** of T if $b, c \in$ T, $\alpha, \beta \in \Gamma$, $a \in$ A implies $b\alpha c\beta a \in$ A, $b\alpha a\beta c \in$ A, $a\alpha b\beta c \in$ A.

Note 2. 9: A nonempty subset A of a ternary Γ-semigroup T is a ternary Γ-ideal of T if and only if it is left ternary Γ-ideal, lateral ternary Γ-ideal and right ternary Γ-ideal of T.

Example 2. 10: Let N be the set of all natural numbers and Γ = 2N. Define the ternary operation from $N \times \Gamma \times N \times \Gamma \times N \to$ N as $(a, \alpha, b, \beta, c) = a.\, \alpha b.\, \beta c$ where '. ' is usual multiplication. Then N is a ternary Γ-semigroup and A = 3N is an ternary Γ-ideal of the ternary Γ-semigroup N.

3. COMPLETELY PRIME TERNARY Γ-IDEAL AND PRIME TERNARY Γ-IDEALS:

We are introduce the notion of a completely prime ideal of a ternary semigroup.

DEFINITION 3. 1: A Ternary Γ-ideal A of a ternary Γ-semigroup T is said to be a ***completely prime* Γ*-ideal*** of T provided $x, y, z \in$ T and $x\Gamma y\Gamma z \subseteq$A implies either $x \in$ A or $y \in$ A or $z \in$ A.

EXAMPLE 3. 2: In the commutative ternary Γ-semigroup Z^- of all negative integers and Γ = { 5k: k ∈ Z^+ }, the ternary Γ-ideal P = { 3k: k ∈ Z^- } is a completely prime ternary Γ-ideal. For $x; y; z \in Z^-$ and $\alpha, \beta \in \Gamma$, $x\alpha y\beta z \in$ P $\Leftrightarrow$ $x\alpha y\beta z$ is divisible by 3 $\Leftrightarrow$ *x* is divisible by 3 or *y* is divisible by 3 or *z* is divisible by 3 $\Leftrightarrow$ $x = 3k_1$ or $y = 3k_2$ or $z = 3k_3$ for $k_1;\ k_2;\ k_3 \in Z^- \Leftrightarrow x \in P$ or $y \in P$ or $z \in P$.

EXAMPLE 3. 3: In example 3. 1. 2., P is a completely prime ternary Γ-ideal. But the ternary Γ-ideal Q = { 30k: k ∈ Z^- } is not a prime Ternary Γ-ideal of Z^-, since (-2)5(-3)10(-5) =-1500 ∈ Q but (-2) ∉ Q, (-3) ∉ Q and (-5) ∉ Q.

THEOREM 3. 4: A Γ-ideal A of a ternary Γ-semigroup T is completely prime if and only if $x_1, x_2, \ldots, x_n \in T$, *n* is odd natural number, $x_1 \Gamma x_2 \Gamma \ldots \Gamma x_n \subseteq A \Rightarrow x_i \in A$ for some *i* = 1, 2, 3, ….. *n*.

***Proof*:** Suppose that A is a completely prime ternary Γ-ideal of T.
Let $x_1, x_2, \ldots, x_n \in T$ where *n* is odd natural number and $x_1 \Gamma x_2 \Gamma \ldots \Gamma x_n \subseteq A$.
If $n = 1$ then clearly $x_1 \in A$.
If $n = 3$ then $x_1 \Gamma x_2 \Gamma x_3 \subseteq A \Rightarrow x_1 \in A$ or $x_2 \in A$ or $x_3 \in A$.
If $n = 5$ then $x_1 \Gamma x_2 \Gamma x_3 \Gamma x_4 \Gamma x_5 \subseteq A \Rightarrow x_1 \Gamma x_2 \Gamma x_3 \subseteq A$ or $x_4 \in A$ or $x_5 \in A$
$\Rightarrow x_1 \in A$ or $x_2 \in A$ or $x_3 \in A$ or $x_4 \in A$ or $x_5 \in A$.
Therefore by induction of *n* is an odd natural number, then $x_1 \Gamma x_2 \Gamma \ldots \Gamma x_n \subseteq A \Rightarrow x_i \in A$ for some *i* = 1, 2, 3, ….. *n*.
The converse part is trivial.
We now introduce the *c*-system of ternary Γ-semigroup.

DEFINITION 3. 5: Let T be a ternary Γ-semigroup. A nonempty subset A of T is said to be a ***c-system*** of T if for each *a, b, c* ∈ A there exists an element α, β ∈ Γ such that $a\alpha b\beta c \in A$.

THEOREM 3. 6: Every ternary Γ-subsemigroup of a ternary Γ-semigroup is a *c-system.*

***Proof*:** Let S be a ternary Γ-subsemigroup of T and *a*, *b*, c ∈ T, α, β ∈ Γ.
Since S is a ternary Γ-subsemigroup of T, $a\alpha b\beta c \in S$ for all α, β ∈ Γ.
Therefore S is a *c*-system.
We now prove a necessary and sufficient condition for a ternary Γ-ideal to be a completely prime ternary Γ-ideal in a ternary Γ-semigroup.

THEOREM 3. 7: A Γ-ideal A of a ternary Γ-semigroup T is completely prime if and only if T\A is either *c*-system of T or empty.

Proof: Suppose that A is a completely prime ternary Γ-ideal l of T and T\A $\neq \emptyset$.
Let *a*, *b*, *c* ∈ T\A. Then $a \notin A$, $b \notin A$, $c \notin A$. Suppose if possible there is no $\alpha, \beta \in \Gamma$ such that $a\alpha b\beta c \notin$ T\A. Then $a\Gamma b\Gamma c \subseteq A$. Since A is completely prime, either $a \in A$ or $b \in A$ or $c \in A$. It is a contradiction. Therefore, there exist elements $\alpha, \beta \in \Gamma$ such that $a\alpha b\beta c \in$ T\A. Hence T\A is a *c*-system of T.
Conversely suppose that T\A is a *c*-system of T or T\A is empty.
If T\A is empty then A = T and hence A is completely prime.
Assume that T\A is a *c*-system of T. Let *a, b, c* ∈ T and $a\Gamma b\Gamma c \subseteq A$.
Suppose if possible $a \notin A$, $b \notin A$, and $c \notin A$.

Then $a \in T\backslash A$, $b \in T\backslash A$ and $c \in T\backslash A$. Since T\A is a *c*-system, then there exist $\alpha, \beta \in \Gamma$ such that $a\alpha b\beta c \in T\backslash A$ and hence $a\alpha b\beta c \notin A$ and hence $a\Gamma b\Gamma c \not\subseteq A$. It is a contradiction. Hence either $a \in A$ or $b \in A$ or $c \in A$. Therefore A is a completely prime ternary Γ-ideal of T.

We now introduce the notion of a prime ternary Γ-ideal of a ternary Γ-semigroup.

DEFINITION 3. 8: A ternary Γ-ideal A of a ternary Γ-semigroup T is said to be a ***prime* Γ*-ideal*** of T provided X, Y, Z are Ternary Γ-ideals of T and $X\Gamma Y\Gamma Z \subseteq A \Rightarrow X \subseteq A$ or $Y \subseteq A$ or $Z \subseteq A$.

THEOREM 3. 9: In a ternary Γ-semigroup T, the following conditions are equivalent:

(i) A is a prime ternary Γ-ideal of T.

(ii) For $a, b, c \in T$; $< a >\Gamma < b >\Gamma < c > \subseteq A$ implies $a \in A$ or $b \in A$ or $c \in A$.

(iii) For $a; b; c \in T$; $T^1\Gamma T^1\Gamma a\Gamma\ T^1\Gamma\ T^1\Gamma b\Gamma\ T^1\ \Gamma T^1\ \Gamma c\ \Gamma T^1\ \Gamma T^1 \subseteq A$ implies $a \in A$ or $b \in A$ or $c \in A$.

Proof: (i) ⇒ (ii): Suppose that A is a prime ternary Γ-ideal of T. Then (i) ⇒ (ii) is obvious.

(ii) ⇒ (iii): Let $a, b, c \in T$ such that $T^1\Gamma T^1\Gamma a\Gamma\ T^1\Gamma\ T^1\Gamma b\Gamma\ T^1\ \Gamma T^1\ \Gamma c\ \Gamma T^1\ \Gamma T^1 \subseteq A$.

Now $< a >\Gamma< b >\Gamma< c > = (T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1)\Gamma(T^1\Gamma T^1\Gamma b\Gamma T^1\Gamma T^1)(T^1\Gamma T^1\Gamma c\Gamma T^1\Gamma T^1) \subseteq T^1 \Gamma T^1\Gamma a\ \Gamma T^1\Gamma T^1\Gamma b\Gamma\ T^1\Gamma T^1\ \Gamma c\Gamma T^1\Gamma T^1 \subseteq A \Rightarrow a \in A$ or $b \in A$ or $c \in A$.

(iii) ⇒ (i): Suppose that $a, b, c \in T$; $T^1T^1aT^1T^1bT^1T^1cT^1T^1 \subseteq A \Rightarrow a \in A$ or $b \in A$ or $c \in A$.

Let X, Y, Z be the three ternary Γ-ideal of T and $X\Gamma Y\Gamma Z \subseteq A$.

Suppose if possible $X \not\subseteq A$, $Y \not\subseteq A$, $Z \not\subseteq A$.

$X \not\subseteq A$, $Y \not\subseteq A$, $Z \not\subseteq A$, there exists a, b, c such that $a \in X$ and $a \notin A$, $b \in Y$ and $b \notin A$ and $c \in Z$ and $c \notin A$. $a \in X$, $b \in Y$, $c \in Z \Rightarrow a\Gamma b\Gamma c \in X\Gamma Y\Gamma Z \subseteq A$.

Now $T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1\Gamma b\Gamma T^1\Gamma T^1\Gamma c\Gamma T^1\Gamma T^1 \subseteq X\Gamma Y\Gamma Z \subseteq A \Rightarrow a \in A$ or $b \in A$ or $c \in A$. It is a contradiction. Therefore $X \subseteq A$ or $Y \subseteq A$ or $Z \subseteq A$ and hence A is a prime ternary Γ-ideal of T.

THEOREM 3. 10: A Γ-ideal A of a ternary Γ-semigroup T is prime if and only if $X_1, X_2, \ldots, X_n \subseteq T$, *n* is odd natural number, $X_1\ \Gamma X_2\Gamma \ldots\ \Gamma X_n \subseteq A \Rightarrow X_i \subseteq A$ for some $i = 1, 2, 3, \ldots n$.

Proof: We can prove this by using mathematical induction as in 3. 1. 4.

The converse part is trivial.

THEOREM 3. 11: Every completely prime ternary Γ-ideal of a ternaryΓ-semigroup T is a prime ternary Γ-ideal of T.

Proof: Suppose that A is a completely prime ideal of a ternary Γ-semigroup T.

Let $a, b, c \in$ T and $< a >\Gamma< b >\Gamma< c > \subseteq$ A. Then $a\Gamma b\Gamma c \subseteq$ A. Since A is a completely prime, either $a \in$ A or $b \in$ A or $c \in$ A. Therefore A is a prime ternary Γ-ideal of T.

THEOREM 3. 12: Let T be a commutative ternary Γ-semigroup. A ternary Γ-ideal P of T is a prime ternary Γ-ideal if and only if P is a completely prime ternary Γ-ideal.

***Proof*:** Suppose that A is a prime ternary Γ-ideal of a ternary Γ-semigroup T.
Let $x, y, z \in$ T and $x\Gamma y\Gamma z \subseteq$ A. Now $x\Gamma y\Gamma z \subseteq$ A, A is a Ternary Γ-ideal $\Rightarrow T^1\Gamma\ T\Gamma T^1\Gamma\ T^1\Gamma(x\Gamma y\Gamma z)\Gamma T^1\Gamma\ T^1\Gamma T^1\Gamma\ T^1 \subseteq P$. Since S is commutative, $T^1\Gamma T\Gamma x\Gamma T^1\ \Gamma T^1\Gamma\ y\Gamma T^1\Gamma T^1 z\Gamma T^1\Gamma\ T = T^1\Gamma T^1\Gamma T^1\Gamma T^1\Gamma(x\Gamma y\Gamma z)\Gamma T^1\Gamma\ T^1\Gamma T^1\Gamma\ T^1 \subseteq P$.
By theorem 3. 1. 9, either $x \in$ A or $y \in$ A or $z \in$ A.
Hence A is a completely prime Ternary Γ-ideal.
Conversely suppose that A is a completely prime ternary Γ-ideal of T.
By theorem 3. 1. 11, P is a prime ternary Γ-ideal of T.
We now introduce the notion of an *m*-system of a ternary Γ-semigroup.

DEFINITION 3. 13: A nonempty subset A of a ternary Γ-semigroup T is said to be an ***m*-system** provided for any $a, b, c \in$ A there exist $x, y \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a\alpha x\beta b\gamma y\delta c \in$ A.
We now prove a necessary and sufficient condition for a ternary Γ-ideal to be a prime ternary Γ-ideal in a ternary Γ-semigroup.

THEOREM 3. 14: A ternary Γ-ideal A of a ternary Γ-semigroup T is a prime Γ-ideal of T if and only if T\A is an *m*-system of T or empty.

***Proof*:** Suppose that A is a prime Ternary Γ-ideal of a ternary Γ-semigroup T and T\A $\neq \emptyset$. Let $a, b, c \in$ T\A. Then $a \notin$ A, $b \notin$ A, $c \notin$ A.
Suppose if possible there exist no $x, y \in$ T and $\alpha, \beta \in \Gamma$ such that $a\alpha x\beta b\gamma y\delta c \in$ T\A.
Then $a\Gamma T\Gamma b\Gamma T\Gamma c \subseteq$ A. Since A is prime, either $a \in$ A or $b \in$ A or $c \in$ A.
It is a contradiction. Therefore there exist $x, y \in$ T and some $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a\alpha x\beta b\gamma y\delta c \in$ T\A. Hence T\A is an *m*-system.
Conversely suppose that T\A is either an *m*-system of T or T\A = $\emptyset$.
If T\A is empty then A = T and hence A is prime. Assume that T\A is an *m*-system of T.
Let $a, b, c \in$ T and $a\Gamma T\Gamma b\Gamma T\Gamma c \subseteq$ A. Suppose if possible $a \notin$ A, $b \notin$ A, $c \notin$ A.
Then $a, b, c \in$ T\A. Since T\A is an *m*-system, there exists $x, y \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a\alpha x\beta b\gamma y\delta c \in$ T\A. Thus $a\alpha x\beta b\gamma y\delta c \notin$ A and hence $a\Gamma T\Gamma b\Gamma T\Gamma c \nsubseteq$ A.
It is a contradiction. Therefore either $a \in$ A or $b \in$ A or $c \in$ A.
Hence A is a prime ternary Γ-ideal of T.

THEOREM 3. 15: If T is a globally idempotent ternary Γ-semigroup then every maximal Γ-ideal of T is a prime ternary Γ-ideal of T.

Proof: Let M be a maximal Ternary Γ-ideal of T. Let A, B, C be three ternary Γ-ideals of T such that $A\Gamma B\Gamma C \subseteq M$. Suppose if possible $A \not\subseteq M$, $B \not\subseteq M$, $C \not\subseteq M$.
Now $A \not\subseteq M \Rightarrow M \cup A$ is a Ternary Γ-ideal of T and $M \subset M \cup A \subseteq T$.
Since M is a maximal, $M \cup A = T$.
Similarly $B \not\subseteq M \Rightarrow M \cup B = T$, $C \not\subseteq M \Rightarrow M \cup C = T$.
Now $T = T\Gamma T\Gamma T = (M \cup A)\Gamma(M \cup B)\Gamma(M \cup C)$
$= [(M\Gamma M)\cup(M\Gamma B)\cup(A\Gamma M)\cup(A\Gamma B)]\Gamma(M\cup C)$
$= (M\Gamma M\Gamma M)\cup(M\Gamma M\Gamma C)\cup(M\Gamma B\Gamma M)\cup(M\Gamma B\Gamma C)\cup$
$(A\Gamma M\Gamma M)\cup(A\Gamma M\Gamma C)\cup(A\Gamma B\Gamma M)\cup(A\Gamma B\Gamma C) \subseteq M \Rightarrow T \subseteq M$. Thus M = T.
It is a contradiction. Therefore either $A \subseteq M$ or $B \subseteq M$ or $C \subseteq M$. Hence M is a prime.

THEOREM 3. 16: If T is a globally idempotent ternary Γ-semigroup having maximal ternary Γ-ideals then T contains semisimple elements.

***Proof*:** Suppose that T is a globally idempotent ternary Γ-semigroup having maximal ternary Γ-ideals. Let M be a maximal Ternary Γ-ideal of T. Then by theorem 3. 1. 15., M is prime. Now if $a \in T\backslash M$ then $< a > \not\subseteq M$ and $< a >\Gamma< a >\Gamma< a > \not\subseteq M$.
Then $T = M \cup < a > = M \cup < a >\Gamma< a >\Gamma< a >$.
Therefore $a \in < a >\Gamma< a >\Gamma< a >$ and hence $< a > = < a >\Gamma< a >\Gamma< a >$.
Thus a is a semisimple element. Therefore T contains semisimple elements.

4. COMPLETELY SEMIPRIME TERNARY Γ-IDEALS AND SEMIPRIME TERNARY Γ-IDEALS:

We now introduce the notion of a completely semiprime ternary Γ-ideal of a ternary Γ-semigroup.

DEFINITION 4. 1: An ideal A of a ternary semigroup T is said to be a ***completely semiprime ideal*** provided $x \in T$, $(x\Gamma)^{n-1}x \subseteq A$ for some odd natural number $n > 1$ implies $x \in A$.

EXAMPLE 4. 2: In commutative ternary Γ-semigroup Z^- of all negative integers and $\Gamma = \{3k: k \in Z^-\}$, the ideal $Q = \{6k: k \in Z^-\}$ is a semiprime ideal. For $x \in Z^-$, $(x\alpha)^2x \in Q \Leftrightarrow (x\alpha)^2x$ is divisible by 6 $\Leftrightarrow x$ is divisible by 6 $\Leftrightarrow x = 6k_1$ for $k_1 \in Z^- \Leftrightarrow x \in Q$.

THEOREM 4. 3: A Γ-ideal A of a ternary Γ-semigroup T is completely semiprime if and only if $x \in T$, $(x\Gamma)^2x \subseteq A$ implies $x \in A$.

Proof: Suppose that A is a completely semiprime ternary Γ-ideal of T.
Then clearly $x \in T$, $(x\Gamma)^2x = x\Gamma x\Gamma x \subseteq A \Rightarrow x \in A$.
Conversely suppose that $x \in T$, $(x\Gamma)^2x \subseteq A \Rightarrow x \in A$.
We prove that $x \in T$, $(x\Gamma)^{n-1}x \subseteq A$, for some odd natural number $n > 1 \Rightarrow x \in A \longrightarrow (1)$,

by induction on n. Clearly (1) is true for $n = 3$. Assume that (1) is true for $n = k$. i. e., $(x\Gamma)^{k-1}x \subseteq A \Rightarrow x \in A$ for some odd natural number $k > 3$.
Suppose that $(x\Gamma)^{k+1}x \subseteq A$. Then $(x\Gamma)^{k+1}x \subseteq A \Rightarrow (x\Gamma)^{k+1}x.\ (x\Gamma)^{k+1}x.\ (x\Gamma)^{k-5}x \subseteq A$
$\Rightarrow (x\Gamma)^{3k-1}x \subseteq A \Rightarrow [(x\Gamma)^{k-1}x)\Gamma]^2\,[(x\Gamma)^{k-1}x] \subseteq A \Rightarrow (x\Gamma)^{k-1}x \subseteq A \Rightarrow x \in A$.
Therefore $\Rightarrow (x\Gamma)^{k-1}x \subseteq A \Rightarrow x \in A$.
By induction, $(x\Gamma)^{n-1}x \subseteq A$, for some odd natural number $n > 1 \Rightarrow x \in A$.
Therefore A is completely semiprime.

THEOREM 4. 4: If A is a completely semiprime ternary Γ-ideal of a ternary Γ-semigroup T, then $x, y, z \in T$, $x\Gamma y\Gamma z \subseteq A$ implies that $x\Gamma y\Gamma T\Gamma T\Gamma z \subseteq A$, $x\Gamma T\Gamma T\Gamma y\Gamma z \subseteq A$ and $x\Gamma T\Gamma y\Gamma T\Gamma z \subseteq A$.

***Proof*:** Let A be a completely semiprime ternary Γ-ideal of a ternary Γ-semigroup T.
Let $x, y, z \in T$, $x\Gamma y\Gamma z \subseteq A$. Now $x\Gamma y\Gamma z \subseteq A$
$\Rightarrow (z\Gamma x\Gamma y\Gamma)^2(z\Gamma x\Gamma y) = (z\Gamma x\Gamma y)\Gamma(z\Gamma x\Gamma y)\Gamma(z\Gamma x\Gamma y) = z\Gamma(x\Gamma y\Gamma z)\Gamma(x\Gamma y\Gamma z)\Gamma x\Gamma y \subseteq A$.
$(z\Gamma x\Gamma y\Gamma)^2(z\Gamma x\Gamma y) \subseteq A$, A is completely semiprime implies $z\Gamma x\Gamma y \subseteq A$.
Let $s, t \in T$. Consider $(x\Gamma y\Gamma s\Gamma t\Gamma z\Gamma)^2(x\Gamma y\Gamma s\Gamma t\Gamma z) = (x\Gamma y\Gamma s\Gamma t\Gamma z)\Gamma(x\Gamma y\Gamma s\Gamma t\Gamma z)\Gamma(x\Gamma y\Gamma s\Gamma t\Gamma z) = x\Gamma y\Gamma s\Gamma t\Gamma(z\Gamma x\Gamma y)\Gamma s\Gamma t\Gamma(z\Gamma x\Gamma y)\Gamma s\Gamma t\Gamma y \subseteq A$.
$(x\Gamma y\Gamma s\Gamma t\Gamma z\Gamma)^2(x\Gamma y\Gamma s\Gamma t\Gamma z) \subseteq A$, A is completely semiprime implies $x\Gamma y\Gamma s\Gamma t\Gamma z \subseteq A$.
Therefore $x, y, z \in T$, $x\Gamma y\Gamma z \subseteq A \Rightarrow x\Gamma y\Gamma s\Gamma t\Gamma z \subseteq A$ for all $s, t \in T \Rightarrow x\Gamma y\Gamma T\Gamma T\Gamma z \subseteq A$.
Now $x\Gamma y\Gamma z \subseteq A \Rightarrow (y\Gamma z\Gamma x\Gamma)^2(y\Gamma z\Gamma x) = (y\Gamma z\Gamma x)\Gamma(y\Gamma z\Gamma x)\Gamma(y\Gamma z\Gamma x)$
$= y\Gamma z\Gamma(x\Gamma y\Gamma z)\Gamma(x\Gamma y\Gamma z)\Gamma x \subseteq A$.
$(y\Gamma z\Gamma x\Gamma)^2(y\Gamma z\Gamma x) \subseteq A$, A is completely semiprime implies $\Rightarrow y\Gamma z\Gamma x \subseteq A$.
Let $s, t \in T$. Consider $(x\Gamma s\Gamma t\Gamma y\Gamma z\Gamma)^2(x\Gamma s\Gamma t\Gamma y\Gamma z) = (x\Gamma s\Gamma t\Gamma y\Gamma z)\Gamma(x\Gamma s\Gamma t\Gamma y\Gamma z)\Gamma(x\Gamma s\Gamma t\Gamma y\Gamma z) = x\Gamma s\Gamma t\Gamma(y\Gamma z\Gamma x)\Gamma s\Gamma t\Gamma(y\Gamma z\Gamma x)\Gamma s\Gamma t\Gamma y\Gamma z \subseteq A$.
$(x\Gamma s\Gamma t\Gamma y\Gamma z\Gamma)^2(x\Gamma s\Gamma t\Gamma y\Gamma z) \subseteq A$, A is completely semiprime implies $x\Gamma s\Gamma t\Gamma y\Gamma z \subseteq A$.
Therefore $x, y, z \in T$, $x\Gamma s\Gamma t\Gamma y\Gamma z \subseteq A$ for all $s, t \in T \Rightarrow x\Gamma T\Gamma T\Gamma y\Gamma z \subseteq A$.
If $s, t \in T$, then $(x\Gamma s\Gamma y\Gamma t\Gamma z\Gamma)^2(x\Gamma s\Gamma y\Gamma t\Gamma z) = (x\Gamma s\Gamma y\Gamma t\Gamma z)\Gamma(x\Gamma s\Gamma y\Gamma t\Gamma z)\Gamma(x\Gamma s\Gamma y\Gamma t\Gamma z)$
$= x\Gamma s\Gamma y\Gamma t\Gamma[z\Gamma x\Gamma(s\Gamma y\Gamma t)\Gamma(z\Gamma x\Gamma s)\Gamma y]\Gamma t\Gamma z \subseteq A$.
$(x\Gamma s\Gamma y\Gamma t\Gamma z\Gamma)^2(x\Gamma s\Gamma y\Gamma t\Gamma z) \subseteq A$, A is completely semiprime $\Rightarrow x\Gamma s\Gamma y\Gamma t\Gamma z \subseteq A$.
Therefore $x, y, z \in T$, $x\Gamma s\Gamma y\Gamma t\Gamma z \subseteq A$ for all $s, t \in T \Rightarrow x\Gamma T\Gamma y\Gamma T\Gamma z \subseteq A$.

COROLLARY 4. 5: If a Γ-ideal A of a ternary Γ-semigroup T is completely semiprime then $x, y, z \in T$, $x\Gamma y\Gamma z \subseteq A \Rightarrow \langle x \rangle \Gamma \langle y \rangle \Gamma \langle z \rangle \subseteq A$.

THEOREM 4. 6: Every completely prime ternary Γ-ideal of a ternary Γ-semigroup T is a completely semiprime ternary Γ-ideal of T.

Proof: Let A be a completely prime ternary Γ-ideal of a ternary Γ-semigroup T.
Suppose that $x \in T$ and $(x\Gamma)^2 x \subseteq A$. Since A is a completely prime ideal of T, $x \in A$.
Therefore T is a completely semiprime ternary Γ-ideal.

THEOREM 4. 7: Let A be a prime ternary Γ-ideal of a ternary Γ-semigroup T. If A is completely semiprime ternary Γ-ideal of T then A is completely prime.

Proof**:** Let x, y, $z \in$ T and $x\Gamma y\Gamma z \subseteq$ A. Since A is completely semiprime, by corollary 4. 5., $x\Gamma y\Gamma z \subseteq A \Rightarrow < x >\Gamma< y >\Gamma< z > \subseteq A \Rightarrow x \in A$ or $y \in$ A or $z \in$ A and hence A is completely prime.

THEOREM 4. 8: The nonempty intersection of any family of a completely prime ternary Γ-ideal of a ternary Γ-semigroup T is a completely semiprime ternary Γ-ideal of T.

Proof: Straight forward.

We now introduce the notion of a *d*-system of a ternary Γ-semigroup.

DEFINITION 4. 9: Let T be a ternary Γ-semigroup. A non-empty subset A of T is said to be a ***d*****-system** of T if $a \in A \Rightarrow (a\Gamma)^{n-1}a \subseteq A$ for all odd natural number n.

We now prove a necessary and sufficient condition for a ternary Γ-ideal to be a completely semiprime ternary Γ-ideal in a ternary Γ-semigroup.

THEOREM 4. 10: A ternary Γ-ideal A of a ternary Γ-semigroup T is completely semiprime if and only if T\A is a *d*-system of T or empty.

Proof: Similar to 3. 7.

We now introduce the notion of a semiprime ternary Γ-ideal of a ternary Γ-semigroup.

DEFINITION 4. 11: A ternary Γ-ideal A of a ternary Γ-semigroup T is said to be ***semiprime ternary Γ-ideal*** provided X is a ternary Γ-ideal of T and $(X\Gamma)^{n-1}X \subseteq A$ for some odd natural number n implies $X \subseteq A$.

THEOREM 4. 12: A ternary Γ-ideal A of a ternary Γ-semigroup T is semiprime if and only if X is a Γ-ideal of T, $(X\Gamma)^2X \subseteq A$ implies $X \subseteq A$.

Proof**:** Suppose that A is a semiprime ternary Γ-ideal. Then clearly $(X\Gamma)^2X \subseteq A$ $\Rightarrow X \subseteq A$. We can verify the converse part by using induction.

THEOREM 4. 13: Every prime ternary Γ-ideal of a ternary Γ-semigroup T is semiprime.

Proof**:** Suppose that A is a prime ternary Γ-ideal of a ternary Γ-semigroup T. Let X be a ternary Γ-ideal of T such that $(X\Gamma)^2X \subseteq A$. Since A is prime, $X \subseteq A$. Hence A is semiprime.

THEOREM 4. 14: If A is a ternary Γ-ideal of a ternary Γ-semigroup T then the following are equivalent.

1. **A is a semiprime ternary Γ-ideal.**
2. **For $a \in T$; $(< a >\Gamma)^2 < a > \subseteq A$ implies $a \in A$.**
3. **For $a \in T$; $T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1 \subseteq A$ implies $a \in A$.**

Proof: (i) ⇒ (ii): Suppose that A is a semiprime ternary Γ-ideal of T.
Then (i) ⇒ (ii) is obvious.
(ii) ⇒ (iii): Let $a \in T$ such that $T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1\Gamma a\ \Gamma T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1 \subseteq A$.
Now $(< a >\Gamma)^2 < a > = < a >\Gamma < a >\Gamma < a >$
$= (T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1)\Gamma(T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1)\Gamma(T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1)$
$\subseteq T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1 \subseteq A \Rightarrow a \in A$.
(iii) ⇒ (i): Suppose that $a \in T$; $T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1 \subseteq A \Rightarrow a \in A$.
Let X be the a ternary Γ-ideal of T and $(X\Gamma)^2X \subseteq A$.
Suppose if possible $X \not\subseteq A$.
$X \not\subseteq A$ there exists a such that $a \in X$ and $a \notin A$. $a \in X \Rightarrow (a\Gamma)^2a \subseteq (X\Gamma)^2X \subseteq A$.
Now $T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1\Gamma a\Gamma T^1\Gamma T^1 \subseteq (X\Gamma)^2X \subseteq A \Rightarrow a \in A$. It is a contradiction.
Therefore $X \subseteq A$ and hence A is a semiprime ternary Γ-ideal of T.

THEOREM 4. 15: Every completely semiprime ternary Γ-ideal of a ternary Γ-semigroup T is a semiprime ternary Γ-ideal of T.

Proof: Suppose that A is a completely semiprime ternary Γ-ideal of a ternary Γ-semigroup T. Let $a \in T$ and $(<a>\Gamma)^{n-1} < a > \subseteq A$ for some odd natural number n.
Now $a\Gamma a\Gamma a\Gamma \ldots\ \Gamma a$($n$ odd terms) $\subseteq < (a\Gamma)^{n-1}a > \subseteq (<a>\Gamma)^{n-1} < a > \subseteq A \Rightarrow (a\Gamma)^{n-1}a \subseteq A \Rightarrow a \in A \Rightarrow < a > \subseteq A$. Therefore A is a semiprime ternary Γ-ideal of T.

THEOREM 4. 16: Let T be a commutative ternary Γ-semigroup. A ternary Γ-ideal A of T is completely semiprime if and only if it is semiprime.

Proof: Similar to 3. 12.

THEOREM 4. 17: The nonempty intersection of any family of prime ternary Γ-ideals of a ternary Γ-semigroup T is a semiprime ternary Γ-ideal of T.

Proof: Straight forward.
We now introduce the notion of an *n*-system of a ternary Γ-semigroup.

DEFINITION 4. 18: A nonempty subset A of a ternary Γ-semigroup T is said to be an ***n*-system** provided for any $a \in A$ there exist $x, y \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a\alpha x\beta a\gamma y\delta a \in A$.

THEOREM 4. 19: Every *m*-system in a ternary Γ-semigroup T is an *n*-system.

Proof: Let A be *m*-system of a ternary Γ-semigroup T. Let $a \in$ A. Since A is *m*-system, $a \in$ A, there exist $x, y \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a\alpha x\beta a\gamma y\delta a \in$ A. Therefore *A is* an *n*-system of T.

We now prove a necessary and sufficient condition for a ternary Γ-ideal to be a semiprime ideal in a ternary Γ-semigroup.

THEOREM 4. 20: A ternary Γ-ideal Q of a ternary Γ-semigroup T is a semiprime ternary Γ-ideal if and only if T\Q is an *n*-system of T (or) empty.

***Proof*:** Similar to 3. 1. 14.

THEOREM 4. 21: If N is an *n*-system in a ternary Γ-semigroup T and $a \in$ N, then there exist an *m*-system M in T such that $a \in$ M and M $\subseteq$ N.

Proof: We construct a subset M of N as follows. Define $a_1 = a$.
Since $a_1 \in$ N and N is an *n*-system, $a_1\alpha x\beta a_1\gamma y a_1 \in$ N, for some $x, y \in$ T, $\alpha, \beta, \gamma, \delta \in \Gamma$.
Thus $(a_1\Gamma T\Gamma a_1\Gamma T\Gamma a_1) \cap N \neq \emptyset$. Let $a_2 \in (a_1\Gamma T\Gamma a_1\Gamma T\Gamma a_1) \cap N$.
Since $a_2 \in$ N and N is an *n*-system, $(a_2\Gamma T\Gamma a_2\Gamma T\Gamma a_2) \cap N \neq \emptyset$ and so on.
In general, if a_i has been defined with $a_i \in$ N, choose a_{i+1} as an element of $(a_i\Gamma T\Gamma a_i\Gamma T\Gamma a_i)) \cap N$. Let M = { $a_1, a_2, \ldots\ldots, a_i, a_{i+1} \ldots\ldots$ }. Now $a \in$ M and M ⊆ N.
We now show that M is an *m*-system.
Let $a_i, a_j, a_k \in$ M (for $i \leq j \leq k$). Then $a_{k+1} \in a_k\Gamma T\Gamma a_k\Gamma T\Gamma a_k \subseteq a_i\Gamma T\Gamma a_i\Gamma T\Gamma a_j \subseteq a_i\Gamma T\Gamma a_j\Gamma T\Gamma a_k \Rightarrow a_{k+1} = a_i\alpha x\beta a_j\gamma y\delta a_k$, $x, y \in$ T, $\alpha, \beta, \gamma, \delta \in \Gamma$.
But $a_{k+1} \in$ M, so $a_{k+1} = a_i\alpha x\beta a_j\gamma y\delta a_k \in$ M, for $x, y \in$ T $\alpha, \beta, \gamma, \delta \in \Gamma$.
Therefore M is an *m*-system.

5. PRIME Γ-RADICAL AND COMPLETELY PRIME Γ-RADICAL:

We use the following notation.

NOTATION 5. 1: If A is an ideal of a ternary Γ-semigroup T, then we associate the following four types of sets.

A_1 = The intersection of all completely prime ternary Γ-ideals of T containing A.

A_2 = $\{x \in T: (x\Gamma)^{n-1}x \subseteq A$ for some odd natural numbers $n\}$

A_3 = The intersection of all prime ternary Γ-ideals of T containing A.

A_4 = $\{x \in T: (<x>\Gamma)^{n-1} <x> \subseteq$ A for some odd natural number $n\}$

THEOREM 5. 2: If A is a ternary Γ-ideal of a ternary Γ-semigroup T, then A $\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$**.**

Proof: i) A $\subseteq A_4$: Let $x \in$ A. Then $(<x>\Gamma)^0 <x> \subseteq$ A and hence $x \in A_4$.
Therefore A $\subseteq A_4$

ii) $A_4 \subseteq A_3$: Let $x \in A_4$. Then $(< x >\Gamma)^{n-1} < x > \subseteq A$ for some odd natural number *n*.
Let P be any prime ternary Γ-ideal of T containing A.
Then $(< x >\Gamma)^{n-1} < x > \subseteq A$ for some odd natural number $n \Rightarrow (< x >\Gamma)^{n-1} < x > \subseteq P$.
Since P is prime, $< x > \subseteq P$ and hence $x \in P$.
Since this is true for all prime ternary Γ-ideals P containing A, $x \in A_3$. Therefore $A_4 \subseteq A_3$.
iii) $A_3 \subseteq A_2$: Let $x \in A_3$. Suppose if possible $x \notin A_2$.
Then $(x\Gamma)^{n-1}x \nsubseteq A$ for all odd natural number *n*.
Consider $Q = \cup(x\Gamma)^{n-1}x$ for all odd natural number *n*, and $x \in T$.
Let $a, b, c \in Q$. Then $a \in (x\Gamma)^{r-1}x$, $b \in (x\Gamma)^{s-1}x$, $c \in (x\Gamma)^{t-1}x$ for some odd natural numbers *r, s, t*. Therefore $a\Gamma b\Gamma c = (x\Gamma)^{r-1}x\Gamma(x\Gamma)^{s-1}x\Gamma(x\Gamma)^{t-1}x = (x\Gamma)^{r+s+t-1}x \in Q$ and hence Q is a ternary Γ-subsemigroup of T and Q is a *c*-system of T and $x \in Q$.
By theorem 3. 7, P = T\Q is a completely prime Ternary Γ-ideal of T and $x \notin P$.
By theorem 3. 11, P is a prime Ternary Γ-ideal of T and $x \notin P$. Therefore $x \notin A_3$.
It is a contradiction. Therefore $x \in A_2$ and hence $A_3 \subseteq A_2$.
iv) $A_2 \subseteq A_1$: Let $x \in A_2$. Now $x \in A_2 \Rightarrow (x\Gamma)^{n-1}x \subseteq A$ for some odd natural number *n*.
Let P be any completely prime ternary Γ-ideal of T containing A.
Then $(x\Gamma)^{n-1}x \subseteq A \subseteq P \Rightarrow (x\Gamma)^{n-1}x \in P \Rightarrow x \in P$. Therefore $x \in A_1$. Therefore $A_2 \subseteq A_1$.
Hence $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

THEOREM 5. 3: If A is a Γ-ideal of a commutative ternary Γ-semigroup T, then $A_1 = A_2 = A_3 = A_4$

NOTE 5. 4: In an arbitrary ternary Γ-semigroup $A_1 \neq A_2 \neq A_3 \neq A_4$.

EXAMPLE 5. 5: Let T be the free ternary Γ-semigroup generated by two alphabets *a, b, c*. It is clear that A = TΓ*a*Γ*a*Γ*a*ΓT is a Ternary Γ-ideal in T. Since $(a\Gamma)^4 a \subseteq T\Gamma a\Gamma a\Gamma a\Gamma T = A$, We have $a \in A_2$. Evidently $(a\Gamma b\Gamma c\Gamma)^{n-1}a\Gamma b\Gamma c \nsubseteq T\Gamma a\Gamma a\Gamma a\Gamma T$ for all odd natural number *n* and thus $a\Gamma b\Gamma c \nsubseteq A_2$. Thus A_2 is not a ternary Γ-ideal l in T. Therefore $A_1 \neq A_2$ and $A_2 \neq A_3$.
We now introduce prime Γ-radical and complete prime Γ-radical of a ternary Γ-ideal in a ternary Γ-semigroup.

DEFINITION 5. 6: If A is a ternary Γ-ideal of a ternary Γ-semigroup T, then the intersection of all prime ternary Γ-ideals of T containing A is called ***prime Γ-radical*** or simply Γ-***radical*** of A and it is denoted by $\sqrt{A}$ or *rad* A.

DEFINITION 5. 7: If A is a ternary Γ-ideal of a ternary Γ-semigroup T, then the intersection of all completely prime ternary Γ-ideals of T containing A is called ***completely prime Γ-radical*** or simply ***complete Γ-radical*** of A and it is denoted by *c. rad* A.

NOTE 5. 8: If A is a ternary Γ-ideal of a ternary Γ-semigroup T, then *rad* A = A_3, *c. rad* A = A_4 and *rad* A ⊆ *c. rad* A**.**

COROLLARY 5. 9: If $a \in \sqrt{A}$, then there exist a odd positive integer *n* such that $(a\Gamma)^{n-1}a \subseteq$ A.

***Proof*:** By theorem 5. 2, $A_3 \subseteq A_2$ and hence $a \in \sqrt{A} = A_3 \subseteq A_2$.
Therefore $(a\Gamma)^{n-1}a \subseteq$ A for some odd positive integer *n*.

COROLLARY 5. 10: If A is a ternary Γ-ideal of a commutative ternary Γ-semigroup T, then *rad* A = *c. rad* A.

proof: By theorem 5. 3, *rad* A = *c. rad* A.

COROLLARY 5. 11: If A is a ternary Γ-ideal of a ternary Γ-semigroup T then *c. rad* A is a completely semiprime ternary Γ-ideal of T.

proof: By theorem 5. 6, *c. rad* A is a completely semiprime ternary Γ-ideal of T.

THEOREM 5. 12: If A, B and C are any three ternary Γ-ideals of a ternary Γ-semigroup T, then

i) $A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B}$

ii) **if A ∩ B∩ C ≠ ∅ then** $\sqrt{A\Gamma B\Gamma C} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$

iii) $\sqrt{\sqrt{A}} = \sqrt{A}$**.**

proof: i) Suppose that A ⊆ B. If P is a prime ternary Γ-ideal containing B then P is a prime Ternary Γ-ideal containing A. Therefore $\sqrt{A} \subseteq \sqrt{B}$.
ii) Let P be a prime ternary Γ-ideal containing AΓBΓC. Then AΓBΓC ⊆ P
⇒ A ⊆ P or B ⊆ P or C ⊆ P ⇒ $A \cap B \cap C \subseteq$ P.
Therefore P is a prime Ternary Γ-ideal containing A ∩ B ∩ C.
Therefore *rad*($A \cap B \cap C$) ⊆ *rad*(AΓBΓC).
Now let P be a prime Ternary Γ-ideal containing $A \cap B \cap C$.
Then $A \cap B \cap C \subseteq$ P ⇒ AΓBΓC ⊆ $A \cap B \cap C \subseteq$ P ⇒ AΓBΓC ⊆ P.
Hence P is a prime Ternary Γ-ideal containing AΓBΓC.
Therefore *rad* (AΓBΓC) ⊆ *rad*($A \cap B \cap C$).
Therefore *rad*(AΓBΓC) = *rad*($A \cap B \cap C$).
Since $A \cap B \cap C \neq \emptyset$, it is clear that A ∩ B ∩ C is a ternary Γ-ideal in T.
Let $x \in \sqrt{A \cap B \cap C}$.
Then there exists an odd natural number $n \in$ N such that $(x\Gamma)^{n-1}x \subseteq A \cap B \cap C$.
Therefore $(x\Gamma)^{n-1}x \subseteq$ A, $(x\Gamma)^{n-1}x \subseteq$ B and $(x\Gamma)^{n-1}x \subseteq$ C.
It follows that $x \in \sqrt{A}$, $x \in \sqrt{B}$ and $x \in \sqrt{C}$. Therefore $x \in \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$.

Consequently, $x \in \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$ implies that there exists odd natural numbers *n, m, p* ∈ N such that $(x\Gamma)^{n-1}x \subseteq A$, $(x\Gamma)^{m-1}x \subseteq B$ and $(x\Gamma)^{p-1}x \subseteq C$.
Clearly, $(x\Gamma)^{n-1}x\Gamma(x\Gamma)^{m-1}x\Gamma(x\Gamma)^{p-1}x = (x\Gamma)^{nmp-1}x \subseteq A \cap B \cap C$.
Thus $x \in \sqrt{A \cap B \cap C}$. Therefore if $A \cap B \cap C \neq \emptyset$ then $\sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$.
iii) $\sqrt{A}$ = The intersection of all prime ternary Γ-ideals of T containing A.
Now $\sqrt{\sqrt{A}}$ = The intersection of all prime ternary Γ-ideals of T containing $\sqrt{A}$.
= The intersection of all prime ternary Γ-ideals of T containing A = $\sqrt{A}$
Therefore $\sqrt{\sqrt{A}} = \sqrt{A}$.

THEOREM 5. 13: If A, B and C are any three ternary Γ-ideals of a ternary Γ-semigroup T, then
(i) A ⊆ B ⇒ *c. rad* A ⊆ *c. rad* B.
(ii) *c. rad* (AΓBΓC) = *c. rad* (A∩B∩C) = *c. rad* (A) ∩ *c. rad* (B) ∩ *c. rad*(*c*).
(iii) *c. rad* (*c. rad* A) = *c. rad* A.

***Proof*:** Similar to 3. 12.

THEOREM 5. 14: If A is a ternary Γ-ideal of a ternary Γ-semigroup T then $\sqrt{A}$ is a semiprime ternary Γ-ideal of T.

proof: By theorem 4. 17, $\sqrt{A}$ is a semiprime ternary Γ-ideal of T.

THEOREM 5. 15: A Γ-ideal Q of ternary Γ-semigroup T is a semiprime ternary Γ-ideal of T if and only if $\sqrt{Q}$ = Q.

Proof: Suppose that Q is a semiprime ternary Γ-ideal. Clearly Q ⊆ √Q.
Suppose if possible √Q ⊈ Q.
Let *a* ∈ √Q and *a* ∉ Q. Now *a* ∉ Q ⇒ *a* ∈ T\Q and Q is semiprime. By theorem 4. 20, T\Q is an *n*-system. By theorem 3. 2. 21, there exists an *m*-system M such that *a* ∈ M ⊆ T\Q. Q ⊆ T\M and now T\M is a prime ternary Γ-ideal of T, *a* ∉ T\M. It is a contradiction. Therefore √Q ⊆ Q. Hence √Q = Q.
Conversely suppose that Q is a ternary Γ-ideal of T such that √Q = Q.
By corollary 5. 14, √Q is a semiprime ternary Γ-ideal of T. Therefore Q is semiprime.

COROLLARY 5. 16: A ternary Γ-ideal Q of a ternary Γ-semigroup T is a semiprime ternary Γ-ideal if and only if Q is the intersection of all prime ternary Γ-ideal of T contains Q.

***Proof*:** By theorem 5. 15., Q is semiprime iff Q is the intersection of all prime ternary Γ-ideals of T contains Q.

COROLLARY 5. 17: If A is a ternary Γ-ideal of a ternary Γ-semigroup T, then √A is the smallest semiprime Γ-ideal of T containing A.

Proof: We have that √A is the intersection of all prime ternary Γ-ideals containing A in T.
Since intersection of prime ternary Γ-ideals is semiprime, we have √A is semiprime.
Further, let Q be any semiprime ternary Γ-ideal containing A, i. e. $A \subseteq Q$. So $\sqrt{A} \subseteq \sqrt{Q}$.
Since Q is semiprime, By theorem 5. 15, $\sqrt{Q} = Q$. Therefore $\sqrt{A} \subseteq Q$.
Hence √A is the smallest semiprime ternary Γ-ideal of T containing A.

THEOREM 5. 18: If P is a prime ternary Γ-ideal of a ternary Γ-semigroup T, then $\sqrt{((P\Gamma)^{n-1}P)} = P$ for all odd natural numbers *n*.

***Proof*:** We use induction on *n*, to prove $\sqrt{((P\Gamma)^{n-1}P)} = P$. First we prove that $\sqrt{P} = P$.
Since P is a prime Ternary Γ-ideal, $P \subseteq \sqrt{P} \subseteq P \Rightarrow \sqrt{P} = P$.
Assume that $\sqrt{((P\Gamma)^{k-1}P)} = P$ for odd natural number *k* such that $1 \leq k < n$.
Now $\sqrt{((P\Gamma)^{k+1}P)} = \sqrt{((P\Gamma)^{k-1}P\Gamma P\Gamma P)} = \sqrt{((P\Gamma)^{k-1}P \cap P \cap P)} = \sqrt{((P\Gamma)^{n-1}P)} \cap \sqrt{P} \cap \sqrt{P} = \sqrt{P} \cap \sqrt{P} \cap \sqrt{P} = P \cap P \cap P = P$. Therefore $\sqrt{((P\Gamma)^{k+1}P)} = P$.
By induction $\sqrt{((P\Gamma)^{n-1}P)} = P$ for all odd natural number *n*.

THEOREM 5. 19: In a ternary Γ-semigroup T with identity there is a unique maximal ternary Γ-ideal M such that $\sqrt{((M\Gamma)^{n-1}M} = M$ for all odd natural numbers *n*.

Proof: Since T contains identity, T is a globally idempotent ternary Γ-semigroup.
Since M is a maximal ternary Γ-ideal of T, by theorem 3. 15 M is prime.
By theorem 5. 18, $\sqrt{((M\Gamma)^{n-1}M} = M$ for all odd natural numbers *n*.

Theorem 5. 20: If A is a ternary Γ-ideal of a ternary Γ-semigroup T then $\sqrt{A}$ ={*x* ∈ T: every *m*-system of T containing *x* meets A } i. e., $\sqrt{A}$ = $\{x \in T : M(x) \cap A \neq \emptyset\}$.

Proof: Suppose that $x \in \sqrt{A}$. Let M be an m-system containing *x*.
Then T\M is a prime ternary Γ-ideal of T and $x \notin$ T\M. If $M \cap A = \emptyset$ then $A \subseteq$ T\M.
Since T\M is a prime ternary Γ-ideal containing A, $\sqrt{A} \subseteq$ T\M and hence $x \in$ T\M.
It is a contradiction. Therefore $M(x) \cap A \neq \emptyset$. Hence $x \in \ x \in T : M(x) \cap A \neq \emptyset$.
Conversely suppose that $x \in \ x \in T : M(x) \cap A \neq \emptyset$.
Suppose if possible $x \notin \sqrt{A}$. Then there exists a prime ternary Γ-ideal P containing A such that $x \notin$ P. Now T\P is an *m*-system and $x \in$ T\P.
$A \subseteq P \Rightarrow$ T\P $\cap A = \emptyset \Rightarrow x \notin \ x \in T : M(x) \cap A \neq \emptyset$.

It is a contradiction. Therefore $x \in \sqrt{A}$. Thus $\sqrt{A} = \; x \in T : M(x) \bigcap A \neq \varnothing \;$.

CONCLUSION:

In this paper we study about the prime ternary Γ-ideals, completely prime ternary Γ-ideals, semiprime ternary Γ-ideals and completely semiprime ternary Γ-ideals, prime Γ-radicals and generalize all these results in general ternary Γ-semigroups.

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