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Degenerate twisted tangent numbers and polynomials associated with the p-adic integral on $\mathbb{Z}_p$

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Abstract

In this paper, we consider the degenerate twisted tangent numbers and polynomials associated with the p-adic integral on $\mathbb{Z}_p$. We also obtain some explicit formulas for degenerate twisted tangent numbers and polynomials.

AMS subject classification: 11B68, 11S40, 11S80.
Keywords: Tangent numbers and polynomials, degenerate tangent polynomials, twisted tangent numbers and polynomials, degenerate twisted tangent numbers and polynomials.

1. Introduction

L. Carlitz introduced the degenerate Bernoulli polynomials (see [1]). Feng Qi *et al.* [2] introduced the partially degenerate Bernoull polynomials of the first kind in p-adic field. T. Kim studied the Barnes' type multiple degenerate Bernoulli and Euler polynomials (see [3]), Recently, Ryoo introduced the degenerate tangent numbers and tangent polynomials (see [7]). In this paper, we introduce degenerate twisted tangent numbers $T_{n,\zeta}(\lambda)$ and tangent polynomials $T_{n,\zeta}(x,\lambda)$. Let p be a fixed odd prime number. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $\mathbb{C}$ denotes the set of complex numbers.

Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that

$|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \quad \text{(see [3])}. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0), \text{ (see [3])}. \tag{1.2}$$

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations (see [8])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. We also have

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.3}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.4}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x|\lambda)_n \frac{t^n}{n!}. \tag{1.5}$$

Let $T_p = \cup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\zeta | \zeta^{p^N} = 1\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$. For $\zeta \in T_p$, and $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, if we take $g(x) = \phi_\zeta(x)(1+\lambda t)^{2x/\lambda}$ in (1.2), then we easily see that

$$I_{-1}(\phi_\zeta(x)(1+\lambda t)^{2x/\lambda} = \int_{\mathbb{Z}_p} \phi_\zeta(x)(1+\lambda t)^{2x/\lambda} d\mu_{-1}(x) = \frac{2}{\zeta(1+\lambda t)^{2/\lambda}+1}.$$

Let us define the degenerate tangent numbers $\mathcal{T}_n(\lambda)$ and polynomials $\mathcal{T}_n(x, \lambda)$ as follows:

$$\frac{2}{(1+\lambda t)^{2/\lambda}+1} = \sum_{n=0}^{\infty} \mathcal{T}_n(\lambda)\frac{t^n}{n!}, \tag{1.6}$$

$$\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} \mathcal{T}_n(x, \lambda)\frac{t^n}{n!}, \text{ (see [7]).} \tag{1.7}$$

Ryoo [6] defined the twisted tangent polynomials $T_{n,\zeta}(x)$ by means of the generating function

$$\int_{\mathbb{Z}_p} \phi_w(y) e^{(x+2y)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_{n,\zeta}(x)\frac{t^n}{n!}. \tag{1.8}$$

and their values at $x = 0$ are called the twisted tangent numbers and denoted $T_{n,\zeta}$.

Recently, many mathematicians have studied in the area of the q-analogues of the degenerate Bernoulli umbers and polynomials, Euler numbers and polynomials, tangent numbers and polynomials (see [1, 2, 3, 4, 5, 6, 8]). Our aim in this paper is to define degenerate twisted tangent polynomials $T_{n,\zeta}(x, \lambda)$. We investigate some properties which are related to twisted tangent numbers $T_{n,\zeta}(\lambda)$ and polynomials $T_{n,\zeta}(x, \lambda)$.

2. Degenerate twisted tangent polynomials

In this section, we introduce degenerate twisted tangent numbers and polynomials, and we obtain explicit formulas for them. For $\zeta \in T_p$, and $t, \lambda \in \mathbb{Z}_p$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$, if we take $g(x) = \phi_\zeta(x)(1+\lambda t)^{2x/\lambda}$ in (1.2), then we easily see that

$$\int_{\mathbb{Z}_p} \phi_\zeta(x)(1+\lambda t)^{2x/\lambda} d\mu_{-1}(x) = \frac{2}{\zeta(1+\lambda t)^{2/\lambda}+1}.$$

Let us define the degenerate twisted tangent numbers $\mathcal{T}_{n,\zeta}(\lambda)$ and polynomials $\mathcal{T}_{n,\zeta}(x, \lambda)$ as follows:

$$\int_{\mathbb{Z}_p} \phi_\zeta(y)(1+\lambda t)^{2y/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \mathcal{T}_{n,\zeta}(\lambda)\frac{t^n}{n!}, \tag{2.1}$$

$$\int_{\mathbb{Z}_p} \phi_\zeta(y)(1+\lambda t)^{(2y+x)/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \mathcal{T}_{n,\zeta}(x, \lambda)\frac{t^n}{n!}. \tag{2.2}$$

Note that $(1+\lambda t)^{1/\lambda}$ tends to e^t as $\lambda \to 0$. By (2.2) and (1.7), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0}\mathcal{T}_{n,\zeta}(x,\lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \sum_{n=0}^{\infty}T_{n,\zeta}(x)\frac{t^n}{n!}.\end{aligned}$$

Thus, we get

$$\lim_{\lambda\to 0}\mathcal{T}_{n,\zeta}(x,\lambda) = T_{n,\zeta}(x),\ (n\geq 0),$$

where $T_{n,\zeta}(x)$ are the twisted tangent polynomials (see [6]). We note that if $\zeta \to 1$, then $\mathcal{T}_{n,\zeta}(x,\lambda) \to \mathcal{T}_n(x,\lambda)$ are the degenerate tangent polynomials.

From (2.2) and (1.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{T}_{n,\zeta}(x,\lambda)\frac{t^n}{n!} &= \frac{2}{\zeta(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \left(\sum_{m=0}^{\infty}\mathcal{T}_{m,\zeta}(\lambda)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(x|\lambda)_l\frac{t^l}{l!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,\zeta}(\lambda)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$\mathcal{T}_{n,\zeta}(x,\lambda) = \sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,\zeta}(\lambda)(x|\lambda)_{n-l}.$$

By (2.1) and (2.2), we obtain the following Witt's formula.

Theorem 2.2. For $n \in \mathbb{Z}_+$, we have

$$\begin{aligned}&\int_{\mathbb{Z}_p}\phi_\zeta(x)(2x|\lambda)_n d\mu_{-1}(x) = \mathcal{T}_{n,\zeta}(\lambda),\\ &\int_{\mathbb{Z}_p}\phi_\zeta(y)(x+2y|\lambda)_n d\mu_{-1}(y) = \mathcal{T}_{n,\zeta}(x,\lambda).\end{aligned}$$

From (2.1), we can derive the following recurrence relation:

$$
\begin{aligned}
2 &= (\zeta(1+\lambda t)^{2/\lambda}+1)\sum_{n=0}^{\infty}\mathcal{T}_{n,\zeta}(\lambda)\frac{t^n}{n!}\\
&= \zeta(1+\lambda t)^{2/\lambda}\sum_{n=0}^{\infty}\mathcal{T}_{n,\zeta}(\lambda)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\mathcal{T}_{n,\zeta}(\lambda)\frac{t^n}{n!}\\
&= \left(\sum_{l=0}^{\infty}\zeta(2|\lambda)_l\frac{t^l}{l!}\sum_{m=0}^{\infty}\mathcal{T}_{m,\zeta}(\lambda)\frac{t^m}{m!}\right)+\sum_{n=0}^{\infty}\mathcal{T}_{n,\zeta}(\lambda)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\zeta(2|\lambda)_l\mathcal{T}_{n-l,\zeta}(\lambda)+\mathcal{T}_n(\lambda)\right)\frac{t^n}{n!}.
\end{aligned}
\tag{2.4}
$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.4), we have the following theorem.

Theorem 2.3. For $n \in \mathbb{Z}_+$, we have

$$
\zeta\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{n-l,\zeta}(\lambda)(2|\lambda)_l+\mathcal{T}_n(\lambda)=\begin{cases}\dfrac{2}{\zeta+1}, & \text{if } n=0,\\ 0, & \text{if } n\neq 0.\end{cases}
$$

By (2.2), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\zeta\mathcal{T}_{n,\zeta}(x+2,\lambda)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\mathcal{T}_{n,\zeta}(x,\lambda)\frac{t^n}{n!}\\
&= \frac{2\zeta}{\zeta(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{(x+2)/\lambda}+\frac{2}{\zeta(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\
&= 2(1+\lambda t)^{x/\lambda}\\
&= 2\sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}.
\end{aligned}
\tag{2.5}
$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.5), we have the following theorem.

Theorem 2.4. For $n \in \mathbb{Z}_+$, we have

$$
\zeta\mathcal{T}_{n,\zeta}(x+2,\lambda)+\mathcal{T}_{n,\zeta}(x,\lambda)=2(x|\lambda)_n.
$$

By (1.1), we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(\zeta^n\mathcal{T}_{m,\zeta}(2n,\lambda)+\mathcal{T}_{m,\zeta}(\lambda)\right)\frac{t^m}{m!}\\
&=\int_{\mathbb{Z}_p}\zeta^{x+n}(1+\lambda t)^{2(x+n)/\lambda}d\mu_{-1}(x)+(-1)^n\int_{\mathbb{Z}_p}\zeta^{x}(1+\lambda t)^{2x/\lambda}d\mu_{-1}(x)\\
&=2\sum_{l=0}^{n-1}(-1)^{n-1-l}\zeta^l(1+\lambda t)^{2l/\lambda}\\
&=\sum_{m=0}^{\infty}\left(2\sum_{l=0}^{n-1}(-1)^{n-1-l}\zeta^l(2l|\lambda)_m\right)\frac{t^m}{m!}.
\end{aligned}\tag{2.6}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.6), we have the following theorem.

Theorem 2.5. For $m\in\mathbb{Z}_+$, we have

$$\zeta^n\mathcal{T}_{m,\zeta}(2n,\lambda)+\mathcal{T}_{m,\zeta}(\lambda)=2\sum_{l=0}^{n-1}(-1)^{n-1-l}\zeta^l(2l|\lambda)_m.$$

By (2.2), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_{m,\zeta^{-1}}(-x,-\lambda)\frac{t^n}{n!}&=\frac{2}{\zeta^{-1}(1-\lambda t)^{-2/\lambda}+1}(1-\lambda t)^{x/\lambda}\\
&=\frac{2\zeta}{(1-\lambda t)^{2/\lambda}+1}(1-\lambda t)^{(x+2)/\lambda}\\
&=\sum_{n=0}^{\infty}(-1)^n\zeta\mathcal{T}_{m,\zeta}(x+2,\lambda)\frac{t^n}{n!}.
\end{aligned}\tag{2.7}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.7), we have the following theorem.

Theorem 2.6. For $n\in\mathbb{Z}_+$, we have

$$\mathcal{T}_{m,\zeta^{-1}}(-x,-\lambda)=(-1)^n\zeta\mathcal{T}_{m,\zeta}(x+2,\lambda),$$

In particular,

$$\mathcal{T}_{m,\zeta^{-1}}(-\lambda)=(-1)^n\zeta T_{m,\zeta}(2,\lambda),$$

For $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{T}_{m,\zeta}(x,\lambda)\frac{t^n}{n!} &= \frac{2}{\zeta(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} \\ &= \frac{2}{\zeta^d(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{x/\lambda}\sum_{l=0}^{d-1}(-1)^l\zeta^l(1+\lambda t)^{2l/\lambda} \\ &= \sum_{n=0}^{\infty}\left(d^n\sum_{l=0}^{d-1}(-1)^l\zeta^l\mathcal{T}_{n,\zeta^d}\left(\frac{2l+x}{d},\frac{\lambda}{d}\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

By comparing coefficients of $\frac{t^m}{m!}$ in the above equation, we obtain the following theorem:

Theorem 2.7. For $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$ and $n \in \mathbb{Z}_+$, we have

$$\mathcal{T}_{m,\zeta}(x,\lambda) = d^n\sum_{l=0}^{d-1}(-1)^l\zeta^l\mathcal{T}_{n,\zeta^d}\left(\frac{2l+x}{d},\frac{\lambda}{d}\right).$$

In particular,

$$\mathcal{T}_{m,\zeta}(\lambda) = d^n\sum_{l=0}^{d-1}(-1)^l\zeta^l\mathcal{T}_{n,\zeta^d}\left(\frac{2l}{d},\frac{\lambda}{d}\right).$$

From (2.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{T}_{n,\zeta}(x+y,\lambda)\frac{t^n}{n!} &= \frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{(x+y)/\lambda} \\ &= \frac{2}{\zeta(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}(1+\lambda t)^{y/\lambda} \\ &= \left(\sum_{n=0}^{\infty}\mathcal{T}_{m,\zeta}(x,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y|\lambda)_n\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,\zeta}(x,\lambda)(y|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.9}$$

Therefore, by (2.9), we obtain the following theorem.

Theorem 2.8. For $n \in \mathbb{Z}_+$, we have

$$\mathcal{T}_{n,\zeta}(x+y,\lambda) = \sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_{l,\zeta}(x,\lambda)(y|\lambda)_{n-l}.$$

From Theorem 2.8, we note that $\mathcal{T}_{n,\lambda}(x)$ is a Sheffer sequence. By replacing t by $\dfrac{e^{\lambda t}-1}{\lambda}$ in (2.2), we obtain

$$\begin{aligned}\left(\frac{2}{\zeta e^{2t}+1}\right)e^{xt} &= \sum_{n=0}^{\infty}\mathcal{T}_{n,\zeta}(x,\lambda)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^{n}\frac{1}{n!}\\ &= \sum_{n=0}^{\infty}\mathcal{T}_{n,\zeta}(x,\lambda)\lambda^{-n}\sum_{m=n}^{\infty}S_2(m,n)\lambda^{m}\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{T}_{n,\zeta}(x,\lambda)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.10}$$

Thus, by (2.10) and (1.9), we have the following theorem.

Theorem 2.9. For $n \in \mathbb{Z}_+$, we have

$$\mathcal{T}_{m,\zeta}(x) = \sum_{n=0}^{m}\lambda^{m-n}\mathcal{T}_{n,\zeta}(x,\lambda)S_2(m,n).$$

By replacing t by $\log(1+\lambda t)^{1/\lambda}$ in (1.9), we have

$$\begin{aligned}\sum_{n=0}^{\infty}T_{n,\zeta}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^{n}\frac{1}{n!} &= \frac{2}{\zeta(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \sum_{m=0}^{\infty}\mathcal{T}_{n,\zeta}(x,\lambda)\frac{t^m}{m!},\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty}T_{n,\zeta}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^{n}\frac{1}{n!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}T_{n,\zeta}(x)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.12}$$

Thus, by (2.11) and (2.12), we have the following theorem.

Theorem 2.10. For $n \in \mathbb{Z}_+$, we have

$$\mathcal{T}_{n,\zeta}(x,\lambda) = \sum_{n=0}^{m}\lambda^{m-n}T_{n,\zeta}(x)S_1(m,n).$$

Letting $\zeta \to 1$ in Theorem 2.10 gives the theorem

$$\mathcal{T}_{n}(x,\lambda) = \sum_{n=0}^{m}\lambda^{m-n}T_{n}(x)S_1(m,n).$$

which was proved by Ryoo [7, Theorem 2.8].

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