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On the second kind generalized twisted (h, q)-Euler numbers and polynomials

C. S. Ryoo

Department of Mathematics,
Hannam University, Daejeon 306-791, Korea

Abstract

In this paper we introduce the second kind generalized twisted (h, q)-Euler numbers $E_{n,\chi,q,w}^{(h)}$ and polynomials $E_{n,\chi,q,w}^{(h)}(x)$.

AMS subject classification:
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1. Introduction

Euler polynomials posses many interesting properties and arising in many areas of mathematics and physics. Many mathematicians have studied Euler numbers and Euler polynomials (see [1, 3, 5, 7, 8, 9, 10]). We introduce the second kind generalized twisted (h, q)-Euler numbers and polynomials. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I_{-1}(g)=\int_X g(x)d\mu_{-1}(x)=\int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x)=\lim_{N\to\infty}\sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [2].} \tag{1.1}$$

If we take $g_n(x)=g(x+n)$ in (1.1), then we see that

$$I_{-1}(g_n)=(-1)^n I_{-1}(g)+2\sum_{l=0}^{n-1}(-1)^{n-1-l}g(l). \tag{1.2}$$

First, we introduce the second kind Euler numbers and Euler polynomials (see [4]). Ryoo [4] investigated the zeros of the second kind Euler polynomials $E_n(x)$. The second kind Euler numbers E_n are defined by the generating function:

$$F(t)=\frac{2e^t}{e^{2t}+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!} \quad (|t|<\frac{\pi}{2}), \tag{1.3}$$

where we use the technique method notation by replacing E^n by $E_n (n\geq 0)$ symbolically. We consider the second kind Euler polynomials $E_n(x)$ as follows:

$$F(x,t)=\left(\frac{2e^t}{e^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}. \tag{1.4}$$

Note that $E_n(x)=\sum_{k=0}^{n}\binom{n}{k}E_k x^{n-k}$. In the special case $x=0$, we define $E_n(0)=E_n$. In [7], we observed the zeros of the second kind q-Euler polynomials $E_{n,q}(x)$. The second kind q-Euler numbers $E_{n,q}$ are defined by the generating function:

$$F_q(t)=\frac{2e^t}{qe^{2t}+1}=\sum_{n=0}^{\infty}E_{n,q}\frac{t^n}{n!}, \tag{1.5}$$

We consider the second kind q-Euler polynomials $E_{n,q}(x)$ as follows:

$$F_q(x,t)=\left(\frac{2e^t}{qe^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}. \tag{1.6}$$

We assume that $h\in\mathbb{Z}$. Let $T_p=\cup_{N\geq 1}C_{p^N}=\lim_{N\to\infty}C_{p^N}$, where $C_{p^N}=\{w|w^{p^N}=1\}$ is the cyclic group of order p^N. For $w\in T_p$, we denote by $\phi_w:\mathbb{Z}_p\to\mathbb{C}_p$ the locally constant function $x \longmapsto w^x$. In [8], we defined the second kind twisted (h, q)-Euler numbers $E_{n,q,w}^{(h)}$ and polynomials $E_{n,q,w}^{(h)}(x)$.

$$F_{q,w}^{(h)}(t) = \frac{2e^t}{wq^h e^{2t} + 1} = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)} \frac{t^n}{n!}, \tag{1.7}$$

$$F_{q,w}^{(h)}(x, t) = \left(\frac{2e^t}{wq^h e^{2t} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} E_{n,q,w}^{(h)}(x) \frac{t^n}{n!}. \tag{1.8}$$

The purpose of this paper is to construct the second kind generalized twisted (h, q)-Euler polynomials $E_{n,\chi,q,w}^{(h)}(x)$ attached to χ and derive a new l-series which interpolates the second kind generalized twisted (h, q)-Euler polynomials $E_{n,\chi,q,w}^{(h)}(x)$.

2. The second kind generalized twisted (h, q)-Euler numbers and polynomials

In this section, our goal is to give generating functions of the second kind generalized (h, q)-Euler numbers and polynomials. These numbers will be used to prove the analytic continuation of the l-series. Let q be a complex number with $|q| < 1$ and w be the p^N-th root of unity. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then the second kind generalized (h, q)-Euler numbers associated with associated with χ, $E_{n,\chi,q,w}^{(h)}$, are defined by the following generating function

$$F_{\chi,q,w}^{(h)}(t) = \frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} e^{(2a+1)t}}{w^d q^{hd} e^{2dt} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,q,w}^{(h)} \frac{t^n}{n!}. \tag{2.1}$$

We now consider the second kind generalized (h, q)-Euler polynomials associated with χ, $E_{n,\chi,q,w}^{(h)}(x)$, are also defined by

$$F_{\chi,q,w}^{(h)}(x, t) = \left(\frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} e^{(2a+1)t}}{w^d q^{hd} e^{2dt} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} E_{n,\chi,q,w}^{(h)}(x) \frac{t^n}{n!}. \tag{2.2}$$

When $\chi = \chi^0$, above (2.1) and (2.2) will become the corresponding definitions of the second kind twisted (h, q)-Euler numbers $E_{n,q,w}^{(h)}$ and polynomials $E_{n,q,w}^{(h)}$(see [6, 8]).

Since

$$\begin{aligned}
&\frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} e^{(2a+1)t}}{w^d q^{hd} e^{2dt} + 1} e^{xt} \\
&= \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} \left(\frac{2e^{dt} e^{(\frac{2a+1+x-d}{d})dt}}{w^d q^{hd} e^{2dt} + 1}\right) \\
&= \sum_{m=0}^{\infty} \left(d^m \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} E_{m,q^d,w^d}\left(\frac{2a+1+x-d}{d}\right)\right) \frac{t^m}{m!},
\end{aligned}$$

we have the following theorem.

Theorem 2.1. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. Then we have

$$(1)\ E^{(h)}_{n,\chi,q,w}(x) = d^m \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} E_{m,q^d,w^d}\left(\frac{2a+1+x-d}{d}\right),$$

$$(2)\ E^{(h)}_{n,\chi,q,w} = d^m \sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} E_{m,q^d,w^d}\left(\frac{2a+1-d}{d}\right),$$

$$(3)\ E^{(h)}_{n,\chi,q,w}(x) = \sum_{l=0}^{n} \binom{n}{l} E^{(h)}_{l,\chi,q,w} x^{n-l}.$$

For $n \in \mathbb{N}$ with $n \equiv 0 \pmod 2$, we have

$$\frac{-2\sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} e^{(2a+1)t}}{w^d q^{hd} e^{2dt}+1} w^{nd} q^{hnd} e^{2ndt} + \frac{2\sum_{a=0}^{d-1} \chi(a)(-1)^a w^a q^{ha} e^{(2a+1)t}}{w^d q^{hd} e^{2dt}+1}$$
$$= \sum_{m=0}^{\infty}\left(2\sum_{a=0}^{nd-1} \chi(a)(-1)^a w^a q^{ha}(2a+1)^m\right)\frac{t^m}{m!}.$$

By comparing coefficients of $\frac{t^m}{m!}$ in the above equation, we have the following theorem:

Theorem 2.2. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, n a positive even integer, and $m \in \mathbb{N}$. Then we have

$$2\sum_{a=0}^{nd-1} \chi(a)(-1)^a w^a q^{ha}(2a+1)^m = -w^{nd} q^{hnd} E^{(h)}_{m,\chi,q,w}(2nd) + E^{(h)}_{m,\chi,q,w}.$$

Next, we introduce the second kind l-series and two variable l-series.

Definition 2.3. For $s \in \mathbb{C}$, define two variable l-series as

$$l^{(h)}_{q,w}(s,x|\chi) = 2\sum_{m=0}^{\infty} \frac{(-1)^m \chi(m) w^m q^{hm}}{(2m+1+x)^s}.$$

By using (2.2), we easily see that

$$\begin{aligned}
F^{(h)}_{\chi,q,w}(x,t) &= \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a w^a q^{ha} e^{(2a+1)t}}{w^d q^{hd} e^{2dt}+1} e^{xt} \\
&= 2\sum_{a=0}^{d-1}\chi(a)(-1)^a w^a q^{ha} e^{(2a+1+x)t}\sum_{l=0}^{\infty}(-1)^l w^{ld} q^{lhd} e^{2dlt} \\
&= 2\sum_{a=0}^{d-1}\sum_{l=0}^{\infty}\chi(a)(-1)^{a+dl} w^{a+dl} q^{h(a+dl)} e^{(2a+1+x+dl)t} \\
&= 2\sum_{m=0}^{\infty}\chi(m)(-1)^m w^m q^{hm} e^{(2m+1+x)t}.
\end{aligned}$$

Then we have

$$\left(\frac{d}{dt}\right)^k F^{(h)}_{\chi,q,w}(x,t)\Bigg|_{t=0} = 2\sum_{n=0}^{\infty}\chi(n)(-1)^n w^n q^{hn}(2n+1+x)^k, \tag{2.3}$$

and

$$\left(\frac{d}{dt}\right)^k \left(\sum_{n=0}^{\infty} E^{(h)}_{n,\chi,q,w}(x)\frac{t^n}{n!}\right)\Big|_{t=0} = E^{(h)}_{k,\chi,q,w}(x), \text{ for } k \in \mathbb{N}. \tag{2.4}$$

By (2.3), (2.4), we have the following theorem.

Theorem 2.4. For any positive integer k, we have

$$E^{(h)}_{k,\chi,q,w}(x) = l^{(h)}_{q,w}(-k, x|\chi).$$

Definition 2.5. For $s \in \mathbb{C}$, define l-series as

$$l^{(h)}_{q,w}(s \mid \chi) = 2\sum_{m=0}^{\infty}\frac{(-1)^m\chi(m)w^m q^{hm}}{(2m+1)^s}.$$

By simple calculation, we have the following theorem.

Theorem 2.6. For any positive integer k, we have

$$l^{(h)}_{q,w}(-k \mid \chi) = E^{(h)}_{k,\chi,q,w}.$$

3. Witt-type formulae on $\mathbb{Z}_p$ in p-adic number field

Our primary aim in this section is to obtain the Witt-type formulae of the second kind generalized twisted (h, q)-Euler numbers $E^{(h)}_{n,\chi,q,w}$ and polynomials $E^{(h)}_{n,\chi,q,w}(x)$

attached to χ. We assume that $q \in \mathbb{C}_p$ with $|q-1|_p < 1$. Let χ be the primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1(\text{mod } 2)$ and $w \in T_p$. Let $g(y) = \chi(y)\phi_w(y)q^{hy}e^{(2y+1+x)t}$. By (1.1), we derive

$$\begin{aligned} I_1\left(\chi(y)\phi_w(y)q^{hy}e^{(2y+1+x)t}\right) &= \int_X \chi(y)\phi_w(y)q^{hy}e^{(2y+1+x)t}d\mu_{-1}(y) \\ &= \frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a w^a q^{ha}e^{(2a+1)t}}{w^d q^{hd}e^{2dt}+1}e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,\chi,q,w}^{(h)}(x)\frac{t^n}{n!}. \end{aligned} \tag{3.1}$$

By using Taylor series of $e^{(2y+1+x)t}$ in the above equation (3.1), we obtain

$$\sum_{n=0}^{\infty}\left(\int_X \chi(y)\phi_w(y)q^{hy}(2y+1+x)^n d\mu_{-1}(y)\right)\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{n,\chi,q,w}^{(h)}(x)\frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we have the Witt formula for the second kind generalized twisted (h, q)- Euler polynomials attached to χ as follows:

Theorem 3.1. For positive integers n, $w \in T_p$, and $h \in \mathbb{Z}$, we have

$$E_{n,\chi,q,w}^{(h)}(x) = \int_X \chi(y)\phi_w(y)q^{hy}(2y+1+x)^n d\mu_{-1}(y). \tag{3.2}$$

Observe that for $x = 0$, the equation (3.2) reduces to (3.3).

Corollary 3.2. For positive integers n, $w \in T_p$, and $h \in \mathbb{Z}$, we have

$$E_{n,\chi,q,w}^{(h)} = \int_X \chi(y)\phi_w(y)q^{hy}(2y+1)^n d\mu_{-1}(y). \tag{3.3}$$

By (3.1) and (1.2), we have the following theorem:

Theorem 3.3. For positive integers n, $w \in T_p$, and $h \in \mathbb{Z}$, we have

$$w^{nd}q^{hnd}E_{m,\chi,q,w}^{(h)}(2nd) - (-1)^n E_{m,\chi,q,w}^{(h)} = 2\sum_{l=0}^{nd-1}(-1)^{n-1-l}\chi(l)w^l q^{hl}(2l+1)^m.$$

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