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Some identities of degenerating Boole polynomials

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Abstract

The Boole polynomial plays an important role in the area of number theory, algebra and umbral calculus. In this paper, we investigate a new and interesting identites of degenerate Boole polynomial which is derived the symmetry properties of the p-adic fermionic intergrals on $\mathbb{Z}_p$.

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1. Introduction

Let p be an odd prime number. $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. Let $C(\mathbb{Z}_p)$ be the space of continuous function on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [11]).} \tag{1.1}$$

Let $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \text{ (see [3, 7-12]).} \tag{1.2}$$

It is well known that the *Euler polynomials of order* $k(\in \mathbb{N})$ are given by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}, \text{ (see [1-4,14]).} \tag{1.3}$$

When $k = 1$, $E_n(x) = E_n^{(1)}(x)$ are called the *ordinary Euler polynomials*, and in particular, if $x = 0$, $E_n = E_n(0)$ are called the *Euler numbers*.

The Changhee polynomials are defined by D. S. Kim et al which is given by the generating function to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}, \text{ (see [6]).} \tag{1.4}$$

When $x = 0$, $Ch_n = Ch_n(0)$ are called the Changhee numbers.

The *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \ (n \geq 0), \text{ (see [13]).}$$

and the *Stirling numbers of the second kind* is defined by

$$(e^t-1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \text{ (see [5, 7]).}$$

The *Boole polynomials* are defined by the generating function to be

$$\sum_{n=0}^{\infty} Bl_n(x|\lambda)\frac{t^n}{n!} = \frac{1}{\left(1+(1+t)^{\lambda}\right)}(1+t)^x, \text{ (see [15,16]).}$$

When $\lambda = 1$, $2Bl_n(x|1) = Ch_n(x)$ are the Changhee polynomials.

Let us take $f(x) = e^{tx}$. Then, by (1.2), we get

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}. \tag{1.5}$$

From (1.5), we have

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.6}$$

Thus, by comparing the coefficients on the both sides of (1.6), we get

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), \quad \text{where } n \in \mathbb{N}\cup\{0\}. \tag{1.7}$$

Now, we define the degenerate Boole numbers as follows:

$$\frac{1}{1+(1+\log(1+qt)^{\frac{1}{q}})^{\lambda}}(1+\log(1+qt)^{\frac{1}{q}})^{x}=\sum_{n=0}^{\infty}Bl_{n,q}(x|\lambda)\frac{t^n}{n!}. \tag{1.8}$$

Note that $\lim_{q\to 0} Bl_{n,q}(x|\lambda)=Bl_n(x|\lambda)$.

In this paper, we investigate identities of symmetry for the degenerate Boole polynomials which are derived from the symmetric properties of the p-adic integral on $\mathbb{Z}_p$.

2. Identities of symmetry for degenerate Boole polynomials

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$. Let us take $f(x)=(1+\log(1+qt)^{\frac{1}{q}})^{\lambda x}$. From (1.2), we have

$$\begin{aligned}\int_{\mathbb{Z}_p}(1+\log(1+qt)^{\frac{1}{q}})^{\lambda x}d\mu_{-1}(x) &= \frac{2}{1+(1+\log(1+qt)^{\frac{1}{q}})^{\lambda}}\\ &=\sum_{n=0}^{\infty}2Bl_{n,q}(\lambda)\frac{t^n}{n!},\end{aligned} \tag{2.1}$$

where $Bl_{n,q}(0|\lambda)=Bl_{n,q}(\lambda)$ are called the degenerate Boole numbers.

By (2.1), it is easy to show that

$$\int_{\mathbb{Z}_p}(1+\log(1+qt)^{\frac{1}{q}})^{x+\lambda y}d\mu_{-1}(y)=\frac{2}{1+(1+\log(1+qt)^{\frac{1}{q}})^{\lambda}}((1+\log(1+qt)^{\frac{1}{q}})^{x}. \tag{2.2}$$

By (1.6) and (2.2), we get

$$\sum_{n=0}^{m}\int_{\mathbb{Z}_p}(x+\lambda y)_n d\mu_{-1}(y)S_1(m,n)q^{m-n}=2Bl_{n,q}(x|\lambda), \quad \text{where } n\in\mathbb{Z}\cup\{0\}. \tag{2.3}$$

and by replacing t by $\frac{1}{q}(e^{q^2t}-1)$ in (1.8), we get

$$\begin{aligned}\sum_{n=0}^{\infty}2Bl_{n,q}(x|\lambda)(\frac{1}{q}(e^{q^2t}-1))^n\frac{1}{n!} &= \frac{2}{1+(1+qt)^{\lambda}}(1+qt)^{x}\\ =\sum_{n=0}^{\infty}2Bl_n(x|\lambda)\frac{q^nt^n}{n!}.\end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we get

$$Bl_m(x|\lambda)=\sum_{n=0}^{m}Bl_{n,q}(x|\lambda)q^{m-n}S_2(m,n) \tag{2.5}$$

$$2Bl_{m,q}(x|\lambda) = q^m \sum_{n=0}^{m} E_n(\frac{x}{\lambda})(\lambda)^n S_1(m,n), \tag{2.6}$$

where $m \in \mathbb{Z} \cup \{0\}$.

Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$.

Then, by(1.1), we see that

$$\begin{aligned} &\int_{\mathbb{Z}_p} (1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_2j+w_1y} d\mu_{-1}(y) \\ &= \lim_{N\to\infty} \sum_{y=0}^{p^N-1} (1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_2j+w_1y}(-1)^y \\ &= \lim_{N\to\infty} \sum_{i=0}^{w_2-1}\sum_{y=0}^{p^N-1} (1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_2j+w_1(i+w_2y)}(-1)^{i+w_2y}. \end{aligned} \tag{2.7}$$

From (2.5), we have

$$\begin{aligned} &\sum_{i=0}^{w_1-1}(-1)^j \int_{\mathbb{Z}_p} (1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_2j+w_1y} d\mu_{-1}(y) \\ &= \lim_{N\to\infty} \sum_{j=0}^{w_1-1}\sum_{i=0}^{w_2-1}\sum_{y=0}^{p^N-1} (-1)^{i+j+y}(1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2(x+y)+w_2j+w_1i}. \end{aligned} \tag{2.8}$$

By the same method as (2.6), we get

$$\begin{aligned} &\sum_{j=0}^{w_2-1}(-1)^j \int_{\mathbb{Z}_p} (1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_1j+w_2y} d\mu_{-1}(y) \\ &= \lim_{N\to\infty} \sum_{j=0}^{w_2-1}\sum_{i=0}^{w_1-1}\sum_{y=0}^{p^N-1} (-1)^{i+j+y}(1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2(x+y)+w_1j+w_2i}. \end{aligned} \tag{2.9}$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2.1. For $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned} &\sum_{i=0}^{w_1-1}(-1)^j \int_{\mathbb{Z}_p} (1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_2j+w_1y} d\mu_{-1}(y) \\ &= \sum_{j=0}^{w_2-1}(-1)^j \int_{\mathbb{Z}_p} (1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_1j+w_2y} d\mu_{-1}(y). \end{aligned} \tag{2.10}$$

Corollary 2.2. For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned}\sum_{i=0}^{w_1-1}(-1)^j \int_{\mathbb{Z}_p}(w_1w_2x+w_2j+w_1y)_n d\mu_{-1}(y)\\ =\sum_{j=0}^{w_2-1}(-1)^j \int_{\mathbb{Z}_p}(w_1w_2x+w_1j+w_2y)_n d\mu_{-1}(y).\end{aligned} \tag{2.11}$$

Now, we observe that

$$\begin{aligned}&\int_{\mathbb{Z}_p}(1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_1j+w_2y} d\mu_{-1}(y)\\ &=\frac{2}{1+(1+\log(1+qt)^{\frac{1}{q}})^{w_2}}(1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_1j}\\ &=\sum_{n=0}^{\infty}2Bl_{n,q}(w_1w_2x+w_1j|w_2)\frac{t^n}{n!}\end{aligned} \tag{2.12}$$

Thus, by (2.10), we get

$$\begin{aligned}\sum_{i=0}^{w_1-1}(-1)^j \int_{\mathbb{Z}_p}(1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_2j+w_1y} d\mu_{-1}(y)\\ =\sum_{n=0}^{\infty}(2\sum_{j=0}^{w_2-1}(-1)^j Bl_{n,q}(w_1w_2x+w_1j|w_2))\frac{t^n}{n!}.\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\sum_{i=0}^{w_1-1}(-1)^j \int_{\mathbb{Z}_p}(1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_2j+w_1y} d\mu_{-1}(y)\\ =\sum_{n=0}^{\infty}(\sum_{j=0}^{w_2-1}(-1)^j \int_{\mathbb{Z}_p}(w_1w_2x+w_1j+w_2y)_n d\mu_{-1}(y)\frac{t^n}{n!}.\end{aligned} \tag{2.14}$$

From (2.11) and (2.12), we have

$$\begin{aligned}2\sum_{j=0}^{w_2-1}(-1)^j Bl_{n,q}(w_1w_2x+w_1j|w_2))\\ =\sum_{j=0}^{w_2-1}(-1)^j \int_{\mathbb{Z}_p}(w_1w_2x+w_1j+w_2y)_n d\mu_{-1}(y).\end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &2\sum_{j=0}^{w_1-1}(-1)^j Bl_{n,q}(w_1w_2x+w_2j|w_1)) \\ &=\sum_{j=0}^{w_1-1}(-1)^j\int_{\mathbb{Z}_p}(w_1w_2x+w_2j+w_1y)_n d\mu_{-1}(y). \end{aligned} \tag{2.16}$$

Therefore, by Corollary 2.2, (2.13) and (2.14), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$,$w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned} &\sum_{j=0}^{w_1-1}(-1)^j Bl_{n,q}(w_1w_2x+w_2j|w_1)) \\ &=\sum_{j=0}^{w_2-1}(-1)^j Bl_{n,q}(w_1w_2x+w_1j|w_2)) \end{aligned} \tag{2.17}$$

Now, we observe that

$$\begin{aligned} &\sum_{j=0}^{w_1-1}(-1)^j\int_{\mathbb{Z}_p}(1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_2j+w_1y}d\mu_{-1}(y) \\ &=\sum_{j=0}^{w_1-1}(-1)^j\sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}(w_1w_2x+w_2j+w_1y)_m d\mu_{-1}(y)\frac{1}{m!}\frac{1}{q^m}(\log(1+qt))^m \\ &=\sum_{j=0}^{w_1-1}(-1)^j\sum_{m=0}^{\infty}Bl_m(w_1w_2x+w_2j|w_1)q^{-m}\sum_{n=m}^{\infty}S_1(n,m)\frac{(qt)^n}{n!} \\ &=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}(-1)^j\sum_{m=0}^{n}Bl_m(w_1w_2x+w_2j|w_1)q^{n-m}S_1(n,m)\right)\frac{(t)^n}{n!}. \end{aligned} \tag{2.18}$$

By the same method of (2.18), we get

$$\begin{aligned} &\sum_{j=0}^{w_2-1}(-1)^j\int_{\mathbb{Z}_p}(1+\log(1+qt)^{\frac{1}{q}})^{w_1w_2x+w_1j+w_2y}d\mu_{-1}(y) \\ &=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}(-1)^j\sum_{m=0}^{n}Bl_m(w_1w_2x+w_1j|w_2)q^{n-m}S_1(n,m)\right)\frac{(t)^n}{n!}. \end{aligned} \tag{2.19}$$

Therefore, by Theorem 2.1, (2.18)and (2.19), we obtain the following theorem.

Theorem 2.4. For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$,$w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^j\sum_{m=0}^{n} Bl_m(w_1w_2x + w_2j|w_1)q^{n-m}S_1(n,m)\\ &= \sum_{j=0}^{w_2-1}(-1)^j\sum_{m=0}^{n} Bl_m(w_1w_2x + w_1j|w_2)q^{n-m}S_1(n,m).\end{aligned} \tag{2.20}$$

Note that

$$\begin{aligned}&Bl_m(w_1w_2x + w_2j|w_1)\\ &= \int_{\mathbb{Z}_p}(w_1w_2x + w_2j + w_1y)_m d\mu_{-1}(y)\\ &= \sum_{l=0}^{m} S_1(m,l)\int_{\mathbb{Z}_p}(w_1w_2x + w_2j + w_1y)^l d\mu_{-1}(y)\\ &= \sum_{l=0}^{m} S_1(m,l)w_1^l\int_{\mathbb{Z}_p}(w_2x + \frac{w_2}{w_1}j + y)^l d\mu_{-1}(y)\\ &= \sum_{l=0}^{m} S_1(m,l)w_1^l E_l(w_2x + \frac{w_2}{w_1}j).\end{aligned} \tag{2.21}$$

Hence by (2.21), we obtain the following theorem.

Theorem 2.5. For $n \geq 0$, $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$,$w_2 \equiv 1 \pmod 2$, we have

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^j\sum_{m=0}^{n}\sum_{l=0}^{m} S_1(m,l)S_1(n,m)q^{n-m}w_1^l E_l(w_2x + \frac{w_2}{w_1}j)\\ &= \sum_{j=0}^{w_2-1}(-1)^j\sum_{m=0}^{n}\sum_{l=0}^{m} S_1(m,l)S_1(n,m)q^{n-m}w_2^l E_l(w_1x + \frac{w_1}{w_2}j).\end{aligned} \tag{2.22}$$

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