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Numerical Critical Value for a Parabolic Equation Modeling Electrostatic Mems

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Abstract.

In this paper, we study the semidiscrete approximation for the following initial-boundary value problem $\begin{cases} u_t(x,t) = u_{xx}(x,t) + \lambda f(x)(1-u(x,t))^{-p}, -l < x < l, t > 0, \\ u(-l,t) = 0, \quad u(l,t) = 0, \ t > 0, \\ u(x,0) = u_0(x) \geq 0, \ -l < x < l, \end{cases}$

where p>1, λ>0 and $f(x) \in C^1([-l,l])$, symmetric and nondecreasing on the interval $(-l,0)$, $0 < f(x) \leq 1$, $f(-l) = 0$, $f(l) = 0$ and $l = \frac{1}{2}$. We determine the critical value of a semidiscrete form of above problem. We also show that the semidiscrete quenching time in certain cases converges to the real one when the mesh size tends to zero. Finally, we give some numerical experiments to illustrate our analysis.

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I. INTRODUCTION

We consider the following initial-boundary value problem

$$u_t(x,t) = u_{xx}(x,t) + \lambda f(x)(1-u(x,t))^{-p}, -l < x < l, t > 0, \tag{1}$$

$$u(-l,t) = 0, \quad u(l,t) = 0, \ t > 0, \tag{2}$$

$$u(x,0) = u_0(x) \geq 0, \ -l < x < l, \tag{3}$$

where p>1, λ>0 and $f(x)\in C^1([-l,l])$, symmetric and nondecreasing on the interval $(-l,0)$, $0< f(x)\le 1,\ f(-l)=0,\ f(l)=0,\ l=\frac{1}{2}$ and $u_0(x)$ is a function which is bounded and symmetric. In addition, $u_0(x)$ is nondecreasing on the interval $(-l,0)$ and $u_0''(x)+\lambda f(x)(1-u_0(x))^{-p}\ge 0$ *on* $(-l,l)$.

Definition 1.1 We say that the classical solution u of (1)-(3) quenches in a finite time if there exists a finite time T_q such that $\|u(.,t)\|_\infty<1$ for $t\in[0,T_q)$ but $\lim_{t\to T_q}\|u(.,t)\|_\infty=1$, where $\|u(.,t)\|_\infty=\max_{-l\le x\le l}|u(x,t)|$. The time T_q is called the quenching time of the solution u.

The problem (1)-(3) models the dynamic deflection of an elastic membrane in a simple electrostatic Micro-Electromechanical System (MEMS) device. The parameter λ characterizes the relative strengh of the electrostatic and mechanical forces in the system and is given in terms of applied voltage. The function *f(x)* represent the varying dielectric permittivity profile(properties of the membrane) and *u*, the deflection of the membrane. Typically a Micro-Electromechanical System(MEMS) device consists of an elastic membrane held at a constant voltage and suspended above a rigid ground plate placed in series with a fixed voltage source. The voltage difference causes a deflection of the membrane, which in turn generates an elastic field in the region between the plate and the membrane. An important nonlinear phenomenon in electrostatically deflected membranes in the so called "pull-in" instability. For moderate voltages, the system is in the stable operation regime: the membrane approaches a steady state and remains separate from the ground plate. When the voltage is increased beyond a critical-value, there is no longer an equilibrium configuration of the membrane. As a result, the membrane collapses onto the ground plate. This phenomenon is also known as "touchdown" or quenching. The critical value of the voltage required for touchdown to occur is termed the pull-in value. (see [24], [25] and the references therein). MEMS technology find applications in the general domains such as automotive domain(airbag systems), consumer domain(computer peripherals), industrial domain(earthquake detection and gas shutoff), military(equipment for soldiers) and biotechnology(microsystems for high throughput drug screening and selection).

The theoretical analysis of quenching solutions for parabolic equations has been investigated by many authors (see [3], [6], [9], [11], [12],[13], [16], [17], [26] and the references cited therein). Local in time existence and the uniqueness of a classical solution have been proved. In particular, in [13], the authors have considered the problem (1)-(3) on a bounded domain Ω of $\Re^N$ with *p=2*. They have shown that there exists a positive number λ* such that if λ>λ* then any solution of (1)-(3) quenches in a finite time whereas if λ<λ* any solution of (1)-(3) exists globally. The number λ* is called the critical value of the problem (1)-(3). In this paper, we are interesting in the numerical study of the critical value, using a semidiscrete form of the problem (1)-(3). Firstly, we show that the above problem possesses a critical value λ*. In the case where the quenching occurs, we show that the semidiscrete quenching time converges

to the real one when the mesh size goes to zero. One may find in [2], [20]-[22], some results concerning the numerical approximations of quenching time.
The present paper is organized as follows. In the next section, we give some lemmas which will be used throughout the paper. In third section, we show the existence of the critical value. In the fourth section, we prove that in the case where quenching occurs, the semidiscrete quenching time converges to real one when the mesh size goes to zero. Finally, in the last section, we give some numerical results to illustrate our analysis.

II. PROPERTIES OF THE SEMIDISCRETE SCHEME

In this section, we give some lemmas which will be used throughout the paper. Let us begin with the construction of a semidiscrete scheme. Let I be a positive integer, and consider the *grid* $x_i=-l+ih$, $0\leq i\leq I$, where $h=2l/I$. We approximate the solution u of (1)-(3) by the solution $U_h(t)=(U_0(t),U_1(t),\ldots,U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt}=\delta^2U_i(t)+\lambda b_i(1-U_i(t))^{-p}, 1\leq i\leq I-1, t\in\left(0,T_q^h\right), \tag{4}$$

$$U_0(t)=0,\quad U_I(t)=0,\ t\in\left(0,T_q^h\right), \tag{5}$$

$$U_i(0)=\varphi_i\geq 0,\ 0\leq i\leq I, \tag{6}$$

where b_i is an approximation of $f(x_i)$, $0\leq i\leq I$, $b_0=0$, $b_I=0$, $0<b_i\leq 1$, $1\leq i\leq I-1$ and $b_{I-i}=b_i$, $1\leq i\leq I-1$, $\delta^+b_i>0$, $1\leq i\leq E\left[\frac{I}{2}\right]-1$, $E\left[\frac{I}{2}\right]$ is the integer part of the number $\frac{I}{2}$, $\delta^2U_i(t)=\frac{U_{i+1}(t)-2U_i(t)+U_{i-1}(t)}{h^2}, 1\leq i\leq I-1,$ $\varphi_0=0, \varphi_I=0, \varphi_{I-i}=\varphi_i, 0\leq i\leq I, \delta^+\varphi_i>0,$ $0\leq i\leq E\left[\frac{I}{2}\right]-1,\ \delta^+\varphi_i=\frac{\varphi_{i+1}-\varphi_i}{h}.$

Here, $(0,T_q^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty<1$ where $\|U_h(t)\|_\infty=\max_{0\leq i\leq I}|U_i(t)|$.

When the time T_q^h is finite, then we say that the solution $U_h(t)$ of (4)-(6) quenches in a finite time, and the time T_q^h is called the quenching time of the solution $U_h(t)$.
The following lemma is a semidiscrete form of the maximum principle.

Lemma 2.1 Let $\zeta_h(t)\in C^0\left([0,T),\Re^{I+1}\right)$ and let $V_h(t)\in C^1\left([0,T),\Re^{I+1}\right)$ be such that

$$\frac{dV_i(t)}{dt}-\delta^2V_i(t)+\zeta_i(t)V_i(t)\geq 0, 1\leq i\leq I-1, t\in(0,T), \tag{7}$$

$$V_0(t)\geq 0,\quad V_I(t)\geq 0, t\in(O,T), \tag{8}$$

$$V_i(0)\geq 0,\quad 0\leq i\leq I. \tag{9}$$

Then $V_i(t)\geq 0$, $0\leq i\leq I$, $t\in(0,T)$.

Proof. Let $T_0 < T$ and introduce the vector $Z_h(t) = e^{\mu t} V_h(t)$ where μ is such that $\zeta_i(t) - \mu > 0$ *for* $t \in [0, T_0]$, $0 \le i \le I$.

Let $m = \min_{0 \le i \le I, 0 \le t \le T_0} Z_i(t)$.

For $i \in \{0, \cdots, I\}$, the function $Z_i(t)$ is continue on the compact $[0, T_0]$. Then there exists $i_0 \in \{0, \cdots, I\}$ and $t_0 \in [0, T_0]$ such that $m = Z_{i_0}(t_0)$.

If $i_0 = 0$ or $i_0 = I$, then $m \ge 0$. If $i_0 \in \{1, \cdots, I-1\}$, we observe that

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \tag{10}$$

$$\delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0. \tag{11}$$

Due to (7), a straightforward computation reveals that

$$\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + (\zeta_{i_0}(t_0) - \mu) Z_{i_0}(t_0) \ge 0\,. \tag{12}$$

It follows from (10)-(11) that $(\zeta_{i_0}(t_0) - \mu) Z_{i_0}(t_0) \ge 0$ which implies that $Z_{i_0}(t_0) \ge 0$ because $\zeta_{i_0}(t_0) - \mu > 0$. We deduce that $V_h(t) \ge 0$ for $t \in [0, T_0]$ and the proof is complete.

Another form of the maximum principle for semidiscrete equations is the comparison lemma below.

Lemma 2.2 Let $g \in C^0(\Re \times \Re, \Re)$. If $V_h(t), W_h(t) \in C^1([0, T], \Re^{I+1})$ are such that

$$\frac{dV_i(t)}{dt} - \delta^2 V_i(t) + g(V_i(t), t) < \frac{dW_i(t)}{dt} - \delta^2 W_i(t) + g(W_i(t), t), \ 1 \le i \le I-1, \ t \in (0, T), \tag{13}$$

$$V_0(t) < W_0(t), \quad V_I(t) < W_I(t), \quad t \in (0, T), \tag{14}$$

$$V_i(0) < W_i(0), \quad 0 \le i \le I, \tag{15}$$

then $V_i(t) < W_i(t)$, $0 \le i \le I$, $t \in (0, T)$.

Proof. Let $Z_h(t) = W_h(t) - V_h(t)$ and let *t0* be the first $t > 0$ such that $Z_i(t) > 0$ for $t \in [0, t_0)$, $0 \le i \le I$, but $Z_{i_0}(t_0) = 0$ for a certain $i_0 \in \{0, \cdots, I\}$.

If $i_0 = 0$ or $i_0 = I$, we have a contradiction because of (14). If $i_0 \in \{1, \cdots, I-1\}$, we obtain

$$\frac{dZ_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{Z_{i_0}(t_0) - Z_{i_0}(t_0 - k)}{k} \le 0, \quad \text{and} \quad \delta^2 Z_{i_0}(t_0) = \frac{Z_{i_0+1}(t_0) - 2Z_{i_0}(t_0) + Z_{i_0-1}(t_0)}{h^2} \ge 0,$$

which implies that $\frac{dZ_{i_0}(t_0)}{dt} - \delta^2 Z_{i_0}(t_0) + g(W_{i_0}(t_0), t_0) - g(V_{i_0}(t_0), t_0) \le 0.$

This inequality contradicts (13) which ends the proof.

The next lemma shows that when *i* is between 1 and *I*-1, then $U_i(t)$ is positive where $U_h(t)$ is the solution of the semidiscrete problem.

Lemma 2.3 Let $U_h(t)$ be the solution of (4)-(6). Then, we have $U_i(t) > 0$ *for* $1 \le i \le I-1, t \in (0, T_q^h)$.

Proof. Assume that there exists a time $t_0 \in (0, T_q^h)$ such that $U_{i_0}(t_0) = 0$ for a certain $i_0 \in \{1, \cdots, I-1\}$. We observe that

$$\frac{dU_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{U_{i_0}(t_0) - U_{i_0}(t_0 - k)}{k} \leq 0, \tag{16}$$

$$\delta^2 U_{i_0}(t_0) = \frac{U_{i_0+1}(t_0) - 2U_{i_0}(t_0) + U_{i_0-1}(t_0)}{h^2} \geq 0, \tag{17}$$

which implies that

$$\frac{dU_{i_0}(t_0)}{dt} - \delta^2 U_{i_0}(t_0) - \lambda b_{i_0}(1 - U_{i_0}(t_0))^{-p} < 0. \tag{18}$$

But this contradicts (4).

Lemma 2.4 Let $U_h(t)$ be the solution of (4)-(6). Then, we have

$$\frac{dU_i(t)}{dt} > 0 \quad for \ 1 \leq i \leq I-1, \ t \in (0, T_q^h). \tag{19}$$

Proof. Setting $W_i(t) = \frac{dU_i(t)}{dt}$, $1 \leq i \leq I-1$, it is not hard to see that

$$\frac{dW_i(t)}{dt} = \delta^2 W_i(t) + \lambda b_i p(1 - U_i(t))^{-p-1} W_i(t),$$

$$1 \leq i \leq I-1, \ t \in (0, T_q^h), \tag{20}$$

$$W_0(t) = 0, \ W_I(t) = 0, \ t \in (0, T_q^h), \tag{21}$$

$$W_i(0) > 0, \ 1 \leq i \leq I-1. \tag{22}$$

Let t_0 be the first $t > 0$ such that $W_{i_0}(t_0) = 0$ for a certain $i_0 \in \{1, \cdots, I-1\}$. Without loss of generality, we may suppose that i_0 is the smallest integer which ensures the equality. We get

$$\frac{dW_{i_0}(t_0)}{dt} = \lim_{k \to 0} \frac{W_{i_0}(t_0) - W_{i_0}(t_0 - k)}{k} \leq 0, \tag{23}$$

$$\delta^2 W_{i_0}(t_0) = \frac{W_{i_0+1}(t_0) - 2W_{i_0}(t_0) + W_{i_0-1}(t_0)}{h^2} \geq 0, \tag{24}$$

which guarantees that

$$\frac{dW_{i_0}(t_0)}{dt} - \delta^2 W_{i_0}(t_0) - \lambda b_{i_0} p(1 - U_{i_0}(t_0))^{-p-1} W_{i_0}(t_0) < 0. \tag{25}$$

Therefore, we have a contradiction because of (20).

The following lemma reveals that the solution $U_h(t)$ of the semidiscrete problem is symmetric and $\delta^+ U_i(t)$ is positive when i is between 1 and $E\left[\frac{I}{2}\right] - 1$.

Lemma 2.5 Let $U_h(t)$ be the solution of (4)-(6). Then, we have for $t \in (0, T_q^h)$

$$U_{I-i}(t) = U_i(t), 0 \leq i \leq I, \delta^+ U_i(t) > 0, 0 \leq i \leq E\left[\frac{I}{2}\right] - 1. \tag{26}$$

Proof. Consider the vector $V_h(t)$ defined as follows $V_i(t)=U_{I-i}(t)$ for $0\le i\le I$.
For $i=0$, we have $V_0(t)=U_{I-0}(t)=U_I(t)=0$, and $i=I$, we also have $V_I(t)=U_{I-I}(t)=U_0(t)=0$. For $i\in\{1,\cdots,I-1\}$, it follows that
$$\frac{dU_{I-i}(t)}{dt}=\delta^2U_{I-i}(t)+\lambda b_{I-i}(1-U_{I-i}(t))^{-p},\quad 1\le i\le I-1,\ t\in\left(0,T_q^h\right).$$
If we replace $U_{I-i}(t)$ by $V_i(t)$ and use the fact that $b_{I-i}(t)=b_i(t)$, we obtain $\frac{dV_i(t)}{dt}-\delta^2V_i(t)=\lambda b_i(1-V_i(t))^{-p},\quad 1\le i\le I-1,\ t\in\left(0,T_q^h\right)$, which implies that $V_h(t)$ is a solution of (4)-(6).
Define the vector $W_h(t)$ such that $W_h(t)=U_h(t)-V_h(t)$.
It is not hard to see that there exists $\theta_i\in(U_i,V_i)$ such that
$$\frac{dW_i(t)}{dt}-\delta^2W_i(t)+p\lambda b_i(1-\theta_i(t))^{-p-1}W_i=0,\ 1\le i\le I-1,\ t\in\left(0,T_q^h\right),$$
$$W_0(t)=0,\quad W_I(t)=0,\quad t\in\left(0,T_q^h\right),\ W_i(0)=0,\quad 0\le i\le I.$$
From Lemma 2.1, it follows that $W_i(t)=0\ for\ 0\le i\le I,\quad t\in\left(0,T_q^h\right)$, which implies that $V_h(t)=U_h(t)$.
Now, define the vector $Z_h(t)$ such that $Z_i(t)=U_{i+1}(t)-U_i(t)>0,\ 0\le i\le E\left[\frac{I}{2}\right]-1$, and let t_0 be the first $t>0$ such that $Z_i(t)>0$ for $t\in[0,t_0)$ but $Z_{i_0}(t_0)=0$. Without loss of the generality, we assume that i_0 is the smallest integer such that $Z_{i_0}(t_0)=0$.
If $i_0=0$, then we have $U_1(t_0)=U_0(t_0)=0$, which is a contradiction because from Lemma 2.3, $U_1(t_0)>0$.
If $i_0=1,\dots,E\left[\frac{I}{2}\right]-2$, we have $\frac{dZ_{i_0}(t_0)}{dt}=\lim_{k\to0}\frac{Z_{i_0}(t_0)-Z_{i_0}(t_0-k)}{k}\le0,\ 1\le i_0\le E\left[\frac{I}{2}\right]-1,$ (27)
and $\delta^2Z_{i_0}(t_0)=\frac{Z_{i_0+1}(t_0)-2Z_{i_0}(t_0)+Z_{i_0-1}(t_0)}{h^2}>0$, which implies that
$$\frac{dZ_{i_0}(t_0)}{dt}-\delta^2Z_{i_0}(t_0)-\lambda b_{i_0+1}(1-U_{i_0+1}(t_0))^{-p}+\lambda b_{i_0}(1-U_{i_0}(t_0))^{-p}<0,$$
But this contradicts (4).
If $i_0=E\left[\frac{I}{2}\right]-1$, we have $U_{i_0+2}(t_0)=U_{E\left[\frac{I}{2}\right]+1}(t_0)=U_{I-E\left[\frac{I}{2}\right]-1}(t_0)$.
- If I is even then $U_{i_0+2}(t_0)=U_{E\left[\frac{I}{2}\right]-1}(t_0)=U_{i_0}(t_0)$, which implies that
$$\delta^2Z_{i_0}(t_0)=\frac{(U_{i_0}-U_{i_0-1})(t_0)}{h^2}=\frac{Z_{i_0-1}(t_0)}{h^2}>0.$$
- If I is odd then $U_{i_0+2}(t_0)=U_{I-E\left[\frac{I-1}{2}\right]-1}(t_0)=U_{E\left[\frac{I+1}{2}\right]-1}(t_0)=U_{i_0+1}(t_0)$, which leads to
$$\delta^2Z_{i_0}(t_0)=\frac{(U_{i_0}-U_{i_0-1})(t_0)}{h^2}=\frac{Z_{i_0-1}(t_0)}{h^2}>0.$$

It is easy to see that $\frac{dZ_{i_0}(t_0)}{dt}-\delta^2 Z_{i_0}(t_0)-\lambda b_{i_0+1}(1-U_{i_0+1}(t_0))^{-p}+\lambda b_{i_0}(1-U_{i_0}(t_0))^{-p}<0,$ which contradicts (4). This ends the proof.
The following lemma is the discrete version of the Green's formula.

Lemma 2.6 Let U_h, $V_h \in \Re^{I+1}$ be two vectors such that $U_0=0, U_I=0, V_0=0, V_I=0.$ Then, we have

$$\sum_{i=1}^{I-1} hU_i\delta^2 V_i=\sum_{i=1}^{I-1} hV_i\delta^2 U_i. \tag{28}$$

Proof. A routine calculation yields $\sum_{i=1}^{I-1} hU_i\delta^2 V_i=\sum_{i=1}^{I-1} hV_i\delta^2 U_i+\frac{V_I U_{I-1}-U_I V_{I-1}+V_0 U_1-U_0 V_1}{h},$ and using the assumptions of the lemma, we obtain the desired result. Now, let us state a result on the operator δ^2.

Lemma 2.7 Let $U_h \in \Re^{I+1}$ be such that $\|U_h\|_\infty<1$ and let p be a positive constant. Then, we have $\delta^2(1-U_i)^{-p}\geq p(1-U_i)^{-p-1}\delta^2 U_i \;\; for\, 1\leq i\leq I-1.$

Proof. Using Taylor's expansion, we get $\delta^2(1-U_i)^{-p}=p(1-U_i)^{-p-1}\delta^2 U_i+\frac{p(p+1)}{2h^2}(U_{i+1}-U_i)^2(1-\chi_i)^{-p-2}+\frac{p(p+1)}{2h^2}(U_{i-1}-U_i)^2(1-\eta_i)^{-p-2} \;\; if\;\; 1\leq i\leq I-1,$
where η_i is an intermediate value between U_{i-1} and U_i and χ_i the one between U_i and U_{i+1}. The result follows taking into account the fact that $\|U_h\|_\infty<1.$
To end this section, let us give another property of the operator δ^2.

Lemma 2.8 Let $U_h, V_h \in \Re^{I+1}$. If $\delta^+(U_i)\delta^+(V_i)\geq 0 \; and \; \delta^-(U_i)\delta^-(V_i)\geq 0, 1\leq i\leq I-1.$ (29)
Then $\delta^2(U_iV_i)\geq U_i\delta^2(V_i)+V_i\delta^2(U_i), 1\leq i\leq I-1,$ where $\delta^+(U_i)=\frac{U_{i+1}-U_i}{h} \; and \; \delta^-(U_i)=\frac{U_{i-1}-U_i}{h}.$

Proof. A straightforward computation yields $h^2\delta^2(U_iV_i)=U_{i+1}V_{i+1}-2U_iV_i+U_{i-1}V_{i-1}$ $=(U_{i+1}-U_i)(V_{i+1}-V_i)+V_i(U_{i+1}-U_i)+U_i(V_{i+1}-V_i)+U_iV_i-2U_iV_i+(U_{i-1}-U_i)(V_{i-1}-V_i)+V_i(U_{i-1}-U_i)$ $+U_i(V_{i-1}-V_i)+U_iV_i,\; 1\leq i\leq I-1,$ which implies that $\delta^2(U_iV_i)=\delta^+(U_i)\delta^+(V_i)+\delta^-(U_i)\delta^-(V_i)+U_i\delta^2(V_i)$ $+V_i\delta^2(U_i), 1\leq i\leq I-1.$ Using (29), we obtain the desired result.

III. NUMERICAL CRITICAL VALUE

In this section, we determine the critical value of the problem (4)-(6). The theorem below, shows that the solution $U_h(t)$ of (4)-(6) quenches in a finite time for λ sufficiently large.

Theorem 3.1 Suppose that $\lambda > \lambda_h \frac{p^p}{b_1(p+1)^{p+1}}$ with $\lambda_h = \frac{2-2\cos(\pi h)}{h^2}$. Then the solution $U_h(t)$ of (4)-(6) quenches in a finite time T_q^h which is estimated as follows $T_q^h \leq \frac{(p+1)^p}{\lambda b_1(p+1)^{p+1} - \lambda_h p^p}$.

Proof. Let $\left(0, T_q^h\right)$ be the maximal time interval on which $\|U_h\|_\infty < 1$. Our aim is to show that T_q^h is finite and satisfies the above inequality. From (4), we observe that $\frac{dU_i(t)}{dt} \geq \delta^2 U_i(t) + \lambda b_1(1-U_i(t))^{-p}, 1 \leq i \leq I-1, t \in \left(0, T_q^h\right)$.

Let a vector $W_h(t)$ such that $\frac{dW_i(t)}{dt} = \delta^2 W_i(t) + \lambda b_1(1-W_i(t))^{-p}, 1 \leq i \leq I-1, t \in \left(0, T_w^h\right)$, $W_0(t) = 0,\ W_I(t) = 0,\ t \in \left(0, T_w^h\right),\ W_i(0) = 0,\ 1 \leq i \leq I-1,$ where T_w^h is the maximal existence time of $W_h(t)$.

Introduce the function $v(t)$ defined as follows $v(t) = \sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h)\sin(i\pi h)W_i(t).$

Take the derivative of v with respect to t and use (4) to obtain $v'(t) = \sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h)\sin(i\pi h)(\delta^2 W_i(t) + \lambda b_1(1-W_i(t))^{-p}.$

We observe that $\delta^2 \sin(i\pi h) = -\lambda_h \sin(i\pi h).$ From the above equality and Lemma 2.6, we arrive at $v'(t) = -\lambda_h v(t) + \lambda b_1 \sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h)\sin(i\pi h)(1-W_i(t))^{-p}.$

By a routine calculation, we find that $\sum_{i=1}^{I-1} \tan(\frac{\pi}{2}h)\sin(i\pi h)$ equals one. Due to the Jensen's Inequality, we get $v'(t) \geq -\lambda_h v(t) + \lambda b_1(1-v(t))^{-p}$.

It is not hard to see that $v(t)(1-v(t))^p$ is bounded from above by $\sup_{0\leq s\leq 1} s(1-s)^p = \frac{p^p}{(p+1)^{p+1}}$. We deduce that $v'(t) \geq (\lambda b_1 - \frac{\lambda_h p^p}{(p+1)^{p+1}})(1-v(t))^{-p}$, which implies that $(1-v(t))^p dv \geq (\lambda b_1 - \frac{\lambda_h p^p}{(p+1)^{p+1}})dt.$

Integrating this inequality over $\left(0, T_w^h\right)$, we find $T_w^h \leq \frac{(p+1)^p}{\lambda b_1(p+1)^{p+1} - \lambda_h p^p}$. The maximum principle implies that $W_i(t) \leq U_i(t), 0 \leq i \leq I, t \in (0, T_0)$ where $T_0 = \min\left\{T_w^h, T_q^h\right\}$. Therefore, we have $T_w^h \geq T_q^h$ and $T_q^h \leq \frac{(p+1)^p}{\lambda b_1(p+1)^{p+1} - \lambda_h p^p}$. Hence, T_q^h is finite and the proof is complete.

Theorem 3.2 Let $U_h(t)$ be the solution of (4)-(6). Then, we have $T_q^h \geq \frac{1}{\lambda(p+1)}$.

Proof. Let i_0 be such that $U_{i_0}(t) = \|U_h(t)\|_\infty$ and $b_{i_0}(t) = b_{E\left[\frac{I}{2}\right]} = 1$. We observe that $\delta^2 U_{i_0}(t) = \frac{U_{i_0+1}(t) - 2U_{i_0}(t) + U_{i_0-1}(t)}{h^2} \leq 0$, which implies that $\frac{dU_{i_0}(t)}{dt} \leq \lambda(1-U_{i_0}(t))^{-p}$, that is to say $(1-U_{i_0}(t))^p dU_{i_0}(t) \leq \lambda dt$.

Integrating this inequality over $(0,T_q^h)$, we get $T_q^h \geq \frac{1}{\lambda(p+1)}$.

Remark 3.1 Theorems 3.1 and 3.2 show that for λ large enough, then the semidiscrete solution quenches in a finite time T_q^h which is bounded from above and below. To obtain the lower bound of the semidiscrete quenching time T_q^h, we consider the following differential equation $\begin{cases} \chi'(t) = \lambda(1-\chi(t))^{-p}, t > 0, \ p > 1, \\ \chi(0) = 0. \end{cases}$

The function $\chi(t)$ quenches in a finite time $T_\chi = \frac{1}{\lambda(p+1)}$. Introduce the vector $V_h(t)$ such that $V_i(t) = \chi(t), 0 \leq i \leq I, t \in (0,T_\chi)$. Setting $Z_h(t) = V_h(t) - U_h(t)$. It is not hard to see that there exists $\omega_i \in (U_i, V_i)$ such that $\frac{dZ_i(t)}{dt} - \delta^2 Z_i(t) + \lambda p b_i (1-\omega_i(t))^{-p-1} Z_i(t) \geq 0$, $1 \leq i \leq I-1, t \in (0,T_1)$, $Z_0(t) \geq 0$, $Z_I(t) \geq 0$, $t \in (0,T_1)$, $Z_i(0) \geq 0$, $0 \leq i \leq I$. where $T_1 = \min\{T_\chi, T_q^h\}$. From Lemma 2.1, it follows that $V_h(t) \geq U_h(t)$ for $0 \leq i \leq I$, $t \in (0,T_1)$.

Therefore, we have $T_\chi \leq T_q^h$ and $T_q^h \geq \frac{1}{\lambda(p+1)}$.

The following result shows that the solution of the semidiscrete problem exists globally for λ sufficiently small.

Theorem 3.3 If $\lambda \leq \frac{8p^p}{b_1(p+1)^{p+1}}$ then the solution $U_h(t)$ of (4)-(6) exists globally and we have $0 \leq U_i(t) \leq \frac{4ih}{p+1}(1-ih), t > 0, 0 \leq i \leq I$.

Proof. Introduce the vector ψ_h defined by $\psi_i = \frac{4ih}{p+1}(1-ih), \ 0 \leq i \leq I$.

We have $\psi_{E\left[\frac{I}{2}\right]}(t) = \frac{1}{p+1}$. It is not hard to see that $\frac{d\psi_i}{dt} - \delta^2 \psi_i = \frac{8}{p+1} \geq \frac{1}{p+1} \frac{\lambda b_1 (p+1)^{p+1}}{p^p} \geq$ $\lambda b_1 (1-\psi_{E\left[\frac{I}{2}\right]})^{-p} \geq \lambda b_1 (1-\psi_i)^{-p}, 1 \leq i \leq I-1$.

Setting $Z_h(t)=\psi_h-U_h(t)$, we find that $\frac{dZ_i(t)}{dt}-\delta^2 Z_i(t)-\lambda p b_i(1-\xi_i(t))^{-p-1}Z_i(t)\geq 0$, $1\leq i\leq I-1, t\in\left(0,T_q^h\right)$, $Z_0(t)=0$, $Z_I(t)=0$, $t\in\left(0,T_q^h\right)$, $Z_i(0)\geq 0$, $0\leq i\leq I$, where $\xi_i(t)$ is an intermediate value between $U_i(t)$ and ψ_i.

From Lemma 2.1, we deduce that $Z_h(t)\geq 0$ for $t\in\left(0,T_q^h\right)$ that is to say $0\leq U_i(t)\leq \frac{4ih}{p+1}(1-ih), 0\leq i\leq I, t\in\left(0,T_q^h\right)$,

This implies that $T_q^h=+\infty$ and the proof is complete.

Theorem 3.4 Assume that there exists two positive numbers λ_1 and λ_2 such that $\lambda_1<\lambda_2$. Then we have $U_h^{\lambda_1}<U_h^{\lambda_2}$ for $t\in\left(0,T_q^{h*}\right)$ where $U_h^{\lambda_i}$ is the solution of (4)-(6) for $\lambda=\lambda_i, i=1,2.$

Proof. Introduce the vector $Z_h(t)=U_h^{\lambda_2}(t)-U_h^{\lambda_1}(t)$. A direct calculation yields $\frac{dZ_i(t)}{dt}-\delta^2 Z_i(t)=\lambda_2 b_i(1-U_i^{\lambda_2})^{-p}-\lambda_1 b_i(1-U_i^{\lambda_1})^{-p}$, $1\leq i\leq I-1, t\in\left(0,T_q^{h*}\right)$, which implies that $\frac{dZ_i(t)}{dt}-\delta^2 Z_i(t)>p\lambda_1 b_i(1-\xi_i^{\lambda})^{-p-1}Z_i(t), 1\leq i\leq I-1, t\in\left(0,T_q^{h*}\right)$, where ξ_i^{λ} is an intermediate value between $U_h^{\lambda_1}$ and $U_h^{\lambda_2}$.

Obviously, we have $Z_0(t)=0, Z_I(t)=0, Z_i(0)=0, 0\leq i\leq I.$

We deduce that $Z_i(t)>0, 1\leq i\leq I-1, t\in\left(0,T_q^{h*}\right)$.

Let $\lambda^*=\{\sup\lambda>0 \textit{ such that the problem } (4)\text{-}(6) \textit{ has global solution}\}$.

From Theorem end, we see that λ^* exists and $\frac{8p^p}{b_1(p+1)^{p+1}}\leq\lambda^*\leq\lambda_h\frac{p^p}{b_1(p+1)^{p+1}}$.

Theorem 3.5 Assume that the solution $U_h(t)$ of (4)-(6) exists globally and is bounded. Then $U_h(t)$ goes to W_h as *t* approaches infinity where W_h is a stationary solution of (4)-(6).

Proof. Introduce the vector $v_h(t)$ such that $v_i(t)=\sum_{j=1}^{I-1}G_{ij}U_j(t)$ where $G_{h,k}$ is the discrete Green function defined by

$$G_{ij}=\begin{cases}\frac{1}{2}ih(1-jh) & if\ 0\leq i\leq j\leq I,\\ \frac{1}{2}jh(1-ih) & if\ 0\leq j\leq i\leq I.\end{cases}$$

A straightforward computation reveals that $\frac{dv_i(t)}{dt}=\sum_{j=1}^{I-1}G_{ij}(\delta^2 U_j(t)+\lambda b_j(1-U_j(t))^{-p})$, which implies that

$$\frac{dv_i(t)}{dt} = -U_i(t) + \lambda \sum_{j=1}^{I-1} G_{ij} b_j (1-U_j(t))^{-p}, \tag{30}$$

From Lemma, $U_j(t)$ is strictly increasing therefore $v_i(t)$ is also strictly increasing. On the other hand, the last term in the right hand side (30) is bounded. Hence, we may conclude that $\frac{dv_i(t)}{dt}$ goes to zero as t approaches infinity.

Setting $\lim_{t\to+\infty} U_i(t) = W_i(t),$ we derive the following equality $W_i = \lambda \sum_{j=1}^{I-1} G_{ij} b_j (1-W_j)^{-p},$ which implies that $\delta^2 W_i + \lambda b_i (1-W_i)^{-p} = 0,\ 1 \le i \le I-1,\ \ W_0 = 0,\ \ W_I = 0,$ and the proof is complete.

IV. CONVERGENCE OF SEMIDISCRETE QUENCHING TIMES

In this section, we show that under some assumptions, the solution $U_h(t)$ of (4)-(6) quenches in a finite time and its quenching time converges to the real one when the mesh size goes to zero.

The following theorem reveals that $U_h(t)$ quenches when λ is sufficiently large.

Theorem 4.1 Suppose that $\lambda > \frac{\pi^2}{b_1(1+p)}$. Then the solution $U_h(t)$ of (4)-(6) quenches in a finite time T_h which satisfies the following estimate $T_q^h \le \frac{-2}{(\pi^2 + \lambda b_1(1+p))} \ln\left(1 - \frac{1}{2}\frac{(\pi^2 + \lambda b_1(1+p))}{(\lambda b_1(1+p))}\right).$

Proof. Since $\left(0, T_q^h\right)$ is the maximal time interval on which $\|U_h(t)\|_\infty < +\infty.$ Our aim is to show that T_q^h is finite and satisfies the above inequality. Introduce the function $J_h(t)$ defined as follows $J_i = \frac{dU_i}{dt} - c_i(1-U_i)^{-p},\ 0 \le i \le I,$ where $c_i(t) = \lambda b_i e^{-\lambda_h t}\sin(i\pi h)$ with $\lambda_h = \frac{2 - 2\cos(\pi h)}{h^2}.$

A direct calculation yields

$$\frac{dJ_i}{dt} - \delta^2 J_i = \frac{d}{dt}\left(\frac{dU_i}{dt} - \delta^2 U_i\right) - \frac{dc_i}{dt}(1-U_i)^{-p} - c_i(1-U_i)^{-p}\frac{dU_i}{dt} + \delta^2\left(c_i(1-U_i)^{-p}\right)$$

We observe that $c_h(t)$ is symmetric and $\delta^+ c_i$ is positive for $0 \le i \le E\left[\frac{I}{2}\right] - 1.$

It follows from Lemma 2.8 and Lemma 2.7 that $\delta^2(c_i(1-U_i)^{-p}) \ge -pc_i(1-U_i)^{-p-1}\delta^2 U_i + (1-U_i)^{-p}\delta^2 c_i.$

Use this inequality to obtain $\frac{dJ_i}{dt} - \delta^2 J_i \ge \frac{d}{dt}\left(\frac{dU_i}{dt} - \delta^2 U_i\right) - c_i(1-U_i)^{-p}\left(\frac{dU_i}{dt} - \delta^2 U_i\right).$

Taking into account (4), we arrive at $\frac{dJ_i}{dt} - \delta^2 J_i \geq \lambda p b_i (1-U_i)^{-p-1} J_i, 1 \leq i \leq I-1$

Obviously, $J_0(t)=0,\ J_I(t)=0$ and $J_h(0) \geq 0$.

Applying Lemma 2.1, we get $J_i(t) \geq 0$, which implies that $\frac{dU_i}{dt} \geq c_i (1-U_i)^{-p},\ 0 \leq i \leq I$.

Let $\alpha = \frac{1}{2}\left(\frac{\lambda b_1 (1+p)}{\pi^2} - 1\right)$. We get $\lambda > \frac{(1+\alpha)\pi^2}{b_1(1+p)}$. Since λ_h goes to π^2 as h tends to zero, then $\lambda_h \leq (1+\alpha)\pi^2$ for h sufficiently small. We deduce that $\frac{dU_{E\left[\frac{I}{2}\right]}}{dt} \geq b_{E\left[\frac{I}{2}\right]} \lambda e^{-(1+\alpha)\pi^2 t} (1-U_{E\left[\frac{I}{2}\right]})^{-p}, t \in \left(0, T_q^h\right)$.

Integrating the above inequality over $\left(0, T_q^h\right)$, we arrive at $\lambda \frac{(1-e^{-(1+\alpha)\pi^2 T_q^h})}{(1+\alpha)\pi^2} \leq \frac{1}{b_1(1+p)}$, which implies that $e^{-(1+\alpha)\pi^2 T_q^h} \geq 1 - \frac{(1+\alpha)\pi^2}{\lambda b_1 (1+p)}$, since $\lambda > \frac{(1+\alpha)\pi^2}{b_1(1+p)}$, then $1 - \frac{(1+\alpha)\pi^2}{\lambda b_1(1+p)} > 0$.

Hence $T_q^h \leq \frac{-1}{(1+\alpha)\pi^2} \ln\left(1 - \frac{(1+\alpha)\pi^2}{\lambda b_1 (1+p)}\right)$.

This implies that T_q^h is finite and we have the desired result.

Remark 4.1 Integrating the above inequality over $\left(0, T_q^h\right)$, and using the fact that $\|U_h(t)\|_\infty = U_k(t)$, we get $T_q^h - t_0 \leq \frac{-1}{(1+\alpha)\pi^2} \ln\left(1 - \frac{(1+\alpha)\pi^2 e^{(1+\alpha)\pi^2 t_0} (1-\|U_h(t_0)\|_\infty)^{p+1}}{\lambda b_1 (1+p)}\right)$, with $\alpha = \frac{1}{2}(\frac{\lambda b_1(1+p)}{\pi^2} - 1)$.

In the following theorem, under adequate hypotheses, we show the convergence of the semidiscrete quenching time to the theoretical one when the mesh size goes to zero. We denote by $u_h(t) = (u(x_0,t),..,u(x_I,t))^T$, $f_h = (f(x_0),.., j(x_I))^T$, $b_h = (b_0,..,b_I)^T$.

In order to prove this result, firstly, we need the following theorem.

Theorem 4.2 Assume that (1)-(3) has a solution $u \in C^{4,1}([-l,l]\times[0,T-\tau])$ such that $\sup_{t\in[0,T-\tau]}\|u(.,t)\|_\infty = \gamma < 1$ with $\tau \in (0,T)$. Suppose that the initial datum at (6) and the varying dielectric permittivity profile at (4) satisfy respectively

$$\|\varphi_h - u_h(0)\|_\infty = o(1) \textit{ and } \|b_h - f_h\|_\infty = o(1) \textit{ as } h \to 0 \tag{31}$$

Then, for h sufficiently small, the problem (4)-(6) has a unique solution $u_h \in C^1(\left[0, T_q^h\right], \Re^{I+1})$ such that

$\max_{0 \leq t \leq T-\tau} \|U_h(t) - u_h(t)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + \|b_h - f_h\|_\infty + h^2)\ as\ h \to 0.$

Proof. Let $K>0$, $L>0$ and $M>0$ such that

$$\frac{\|u_{xxxx}\|_\infty}{12} \le K,\ \lambda p\|b_h\|_\infty(1-\frac{\gamma}{2})^{-p-1} \le M,\ \lambda(1-\frac{\gamma}{2})^{-p} \le L. \qquad (32)$$

The problem (4)-(6) has for each h, a unique solution $u_h \in C^1([0,T_q^h],\Re^{I+1})$. Let $t(h) \le \min\{T-\tau, T_q^h\}$ be the greatest value of $t>0$. There exists a positive real β (with $\gamma<\beta<1$) such that

$$\|U_h(t)-u_h(t)\|_\infty < \frac{\beta-\gamma}{2} \text{ for } t \in (0,t(h)). \qquad (33)$$

From (31), we deduce that $t(h)>0$ for h sufficiently small. By the triangle inequality, we obtain $\|U_h(t)\|_\infty \le \|u(.,t)\|_\infty + \|U_h(t)-u_h(t)\|_\infty$ *for* $t \in (0,t(h))$, which implies that

$$\|U_h(t)\|_\infty \le \gamma + \frac{\beta-\gamma}{2} = \frac{\beta+\gamma}{2} < 1 \text{ for } t \in (0,t(h)). \qquad (34)$$

Let $e_h(t) = U_h(t)-u_h(t)$ be the error of discretization. Using Taylor's expansion, we have for $t \in (0,t(h))$,

$\frac{de_i(t)}{dt} - \delta^2 e_i(t) = \frac{h^2}{12} u_{xxxx}(\tilde{x}_i,t) + \lambda p b_i(1-\theta_i)^{-p-1} e_i(t) + \lambda(b_i - f(x_i))(1-u(x_i,t))^{-p}, 1 \le i \le I-1$, where θ_i is an intermediate value between $U_i(t)$ and $u(x_i,t)$. Using (32) and (34), we arrive at

$$\frac{de_i(t)}{dt} - \delta^2 e_i(t) \le M|e_i(t)| + L\|b_h - f_h\| + Kh^2, 1 \le i \le I-1. \qquad (35)$$

Let $z_h(t)$ the vector defined by

$$z_i(t) = e^{(M+1)t}(\|\varphi_h - u_h(0)\|_\infty + L\|b_h - f_h\|_\infty + Kh^2), 0 \le i \le I.$$

A direct calculation yields $\frac{dz_i(t)}{dt} - \delta^2 z_i(t) > M|z_i(t)| + L\|b_h - f_h\|_\infty + Kh^2$, $1 \le i \le I-1$, $t \in (0,t(h))$, $z_0(t) > e_0(t)$, $z_I(t) > e_I(t)$, $t \in (0,t(h))$, $z_i(0) > e_i(0)$, $0 \le i \le I$.

It follows from Lemma 2.2 that $z_i(t) > e_i(t)$ *for* $t \in (0,t(h)), 0 \le i \le I$. By the same reasoning, we also prove that $z_i(t) > -e_i(t)$ *for* $t \in (0,t(h)), 0 \le i \le I$, which implies that $z_i(t) > |e_i(t)|$ for $t \in (0,t(h))$, $0 \le i \le I$.

We deduce that $\|U_h(t)-u_h(t)\|_\infty \le e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + L\|b_h - f_h\|_\infty + Kh^2)$, $t \in (0,t(h))$.

In order to show that $t(h) = \min\{T-\tau, T_q^h\}$ we argue by contradiction. Suppose that $t(h) < \min\{T-\tau, T_q^h\}$ From (33), we obtain

$$\frac{\beta-\gamma}{2} \le \|U_h(t(h)) - u_h(t(h))\|_\infty \le e^{(M+1)T}(\|\varphi_h - u_h(0)\|_\infty + L\|b_h - f_h\|_\infty + Kh^2) \qquad (36)$$

We remark that when h tends to zero, $\frac{\beta-\gamma}{2} \le 0$, which is impossible. Consequently $t(h) = \min\{T-\tau, T_q^h\}$. Let us show that $t(h) = T-\tau$. Suppose that $t(h) = T_q^h < T-\tau$. Arguing as above, we obtain a contradiction, which leads us to the desired result.

Now, we prove the main result of this section, the convergence of the quenching time.

Theorem 4.3 Suppose that the problem (1)-(3) has a solution u which quenches in a finite time T_q such that $u \in C^{4,1}([-l,l]\times[0,T_q))$ and the initial datum at (6) and the varying dielectric permittivity profile at (4) satisfy the hypothesis (31). Under the assumptions of Theorem 4.1, the problem (4)-(6) has a solution U_h which quenches in a finite time T_q^h and $\lim_{h\to 0} T_q^h = T_q$.

Proof. Let $0<\varepsilon<\frac{T_q}{2}$. There exists $\rho=\beta-\gamma$ $(with\, 0<\gamma<\beta<1)$ such that

$$\frac{-1}{(1+\alpha)\pi^2}\ln\left(1-\frac{(1+\alpha)\pi^2 e^{(1+\alpha)\pi^2 T_q}(1-\eta)^{p+1}}{\lambda b_1(1+p)}\right)\le\frac{\varepsilon}{2},\quad for\ \eta\in[1-\rho,1), \tag{37}$$

with $\alpha=\frac{1}{2}(\frac{\lambda b_1(1+p)}{\pi^2}-1)$. Since $\lim_{t\to T_q}\|u(.,t)\|_\infty=1$, there exist $T_1<T_q$ and $|T_q-T_1|<\frac{\varepsilon}{2}$ such that $1>\|u(.,t)\|_\infty\ge 1-\frac{\rho}{2}$ for $t\in[T_1,T_q)$. From Theorem 4.2, the problem (4)-(6) has for h sufficiently small, the unique solution $U_h(t)$ such that $\|U_h(t)-u_h(t)\|_\infty<\frac{\rho}{2}$ for $t\in[0,T_2]$ where $T_2=\frac{T_1+T_q}{2}$. Using the triangle inequality, we get $\|U_h(t)\|_\infty\ge\|u(.,t)\|_\infty-\|U_h(t)-u_h(t)\|_\infty\ge 1-\frac{\rho}{2}-\frac{\rho}{2}$ $for\ t\in[0,T_2]$, which implies that $\|U_h(t)\|_\infty\ge 1-\rho\ for\ t\in[0,T_2]$.

From Theorem 4.1, $U_h(t)$ quenches at time T_q^h. Using inequality (37) and the Remark 4.1, we arrive at $|T_q^h-T_1|\le\frac{-1}{(1+\alpha)\pi^2}\ln\left(1-\frac{(1+\alpha)\pi^2 e^{(1+\alpha)\pi^2 T_1}(1-\|U_h(T_1)\|_\infty)^{p+1}}{\lambda b_1(1+p)}\right)\le\frac{\varepsilon}{2}$, with $\alpha=\frac{1}{2}(\frac{\lambda b_1(1+p)}{\pi^2}-1)$. It follows that $|T_q^h-T_q|\le|T_q^h-T_1|+|T_1-T_q|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$.

This complete the proof.

V. NUMERICAL RESULTS

In this section, we present some numerical approximations to the quenching time and the critical value of problem (1)-(3) in the case where $u_0(x)=0$ and $f(x)=16\left(x^2-\frac{1}{4}\right)^2$.

Firstly, we consider the following explicit scheme

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n^e}=\frac{U_{i+1}^{(n)}-2U_i^{(n)}+U_{i-1}^{(n)}}{h^2}+\lambda b_i(1-U_i^{(n)})^{-p},\qquad 1\le i\le I-1,\qquad U_0^{(n)}=0,\ U_I^{(n)}=0,$$

$U_i^{(0)}=0,\ 0\le i\le I$, and secondly, we use the following implicit scheme

$$\frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n}=\frac{U_{i+1}^{(n+1)}-2U_i^{(n+1)}+U_{i-1}^{(n+1)}}{h^2}+\lambda b_i(1-U_i^{(n)})^{-p},\ 1\le i\le I-1,\qquad U_0^{(n+1)}=0,\ U_I^{(n+1)}=0,$$

$U_i^{(0)}=0,\ 0\le i\le I$, where $n\ge 0, b_i=16(x_i^2-\frac{1}{4})^2, \Delta t_n=h^2(1-\|U_h^{(n)}\|_\infty)^{p+1}$, $\Delta t_n^e=\min\left\{\frac{h^2}{2},\Delta t_n\right\}$ and $T_\lambda=\sum_{j=0}^{n-1}\Delta t_j$.

In the following tables, we present some numerical results for different conditions on the permittivity profile *f(x)*.

First case : *0<f(x)<1* where *I=16*.

Table 1: Exponent of reaction term, numerical critical value or numerical values for pull-in voltage, applied voltage, numerical quenching time obtain with explicit scheme for any applied voltage and numerical quenching time obtain with implicit scheme for any applied voltage

p	λ^*	λ	T_λ^e	T_λ^i
2	26.532458	27	1.179225	1.193489
3	18.888830	19	1.494632	1.512930
4	14.671388	15	1.305363	1.322107
5	11.995650	12	1.785737	1.808116

Second case : *f(x) = 1* where *I = 16*.

Table 2: Exponent of reaction term, numerical critical value or numerical values for pull-in voltage, applied voltage, numerical quenching time obtain with explicit scheme for any applied voltage and numerical quenching time obtain with implicit scheme for any applied voltage

p	λ^*	λ	T_λ^e	T_λ^i
2	1.457471	2	0.327082	0.332036
3	1.375940	2	0.179801	0.182862
4	0.805923	1	0.481578	0.488721
5	0.658941	1	0.296549	0.301208

In the following, we also give some plots to illustrate our analysis.

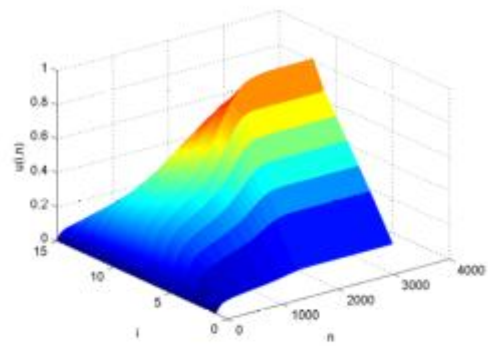

Figure 1: Evolution of the discrete solution for $\lambda = 27$, $p = 2$ and $0 < f(x) < 1$ (Explicit scheme).

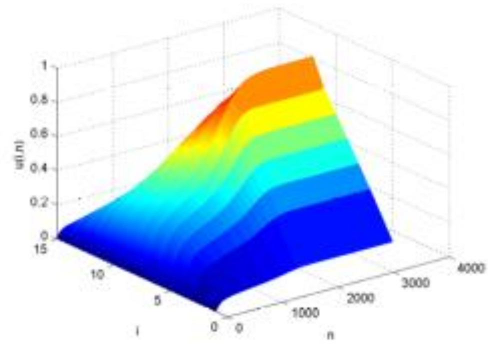

Figure 2: Evolution of the discrete solution for $\lambda = 27$, $p = 2$ and $0 < f(x) < 1$ (Implicit scheme).

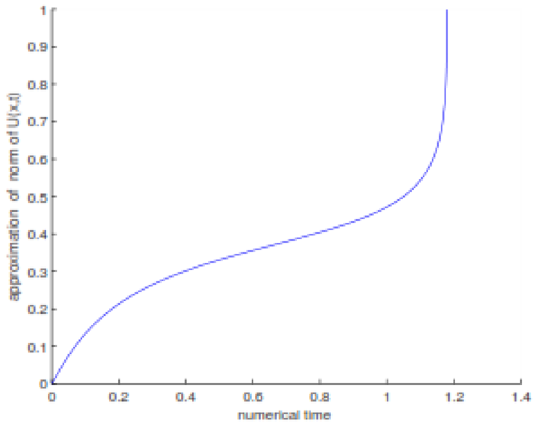

Figure 3: Profil of the approximation of $u(x,T)$ where, T is the quenching time for $\lambda = 27$, $p = 2$ and $0 < f(x) < 1$ (Explicit scheme).

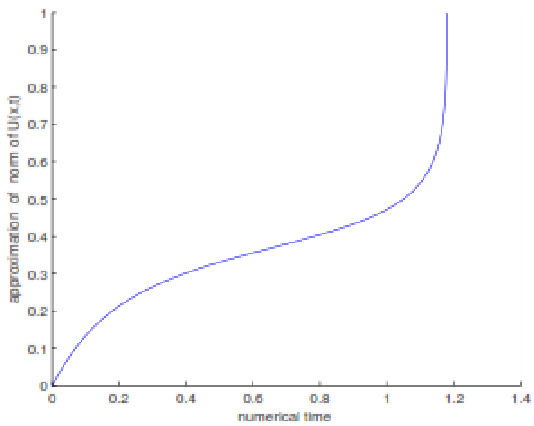

Figure 4: Profil of the approximation of $u(x,T)$ where, T is the quenching time for $\lambda = 27$, $p = 2$ and $0 < f(x) < 1$ (Implicit scheme).

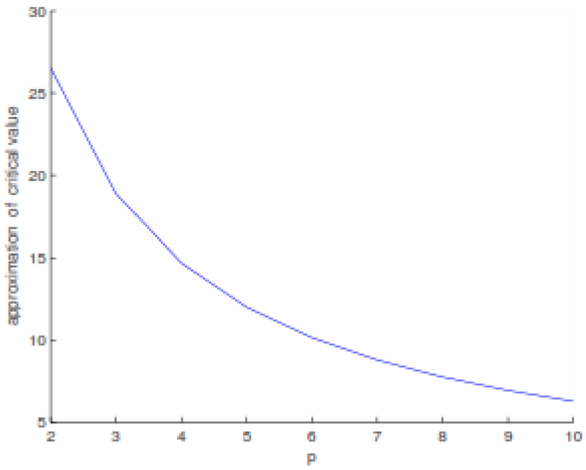

Figure 5: Plot of numerical critical value versus exponent p.

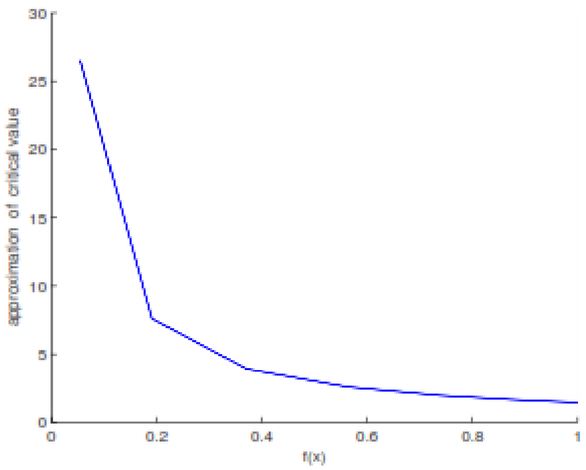

Figure 6: Plot of numerical critical value versus the permittivity profile $f(x)$.

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