Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 6 (2015), pp. 5203–5222

http://www.ripublication.com/gjpam.htm

Some Best Proximity Point and Fixed Point Theorems via Generalized Cyclic Contraction in G-Metric Spaces

Deepak Singh

Department of Applied Sciences,
NITTTR, Under Ministry of HRD,
Govt. of India, Bhopal, (M.P.), India, 462002.
E-mail: dk.singh1002@gmail.com

Vishal Joshi

Department of Applied Mathematics,
Jabalpur Engineering College,
Jabalpur, (M.P.), India
E-mail: joshinvishal76@gmail.com

Jong Kyu Kim

Department of Mathematics Education,
Kyungnam University, Changwon,
Gyeongnam, 631-701, Korea.
E-mail: jongkyuk@kyungnam.ac.kr

Abstract

Very recently, Hussain et al. [1] introduced certain new class of proximal contraction mappings and established the best proximity point theorems in G-metric spaces. In this article, acknowledging the aforesaid concept, some best proximity point theorems are proved under generalized proximal cyclic ϕ-contraction condition which is new in the frame work of G-metric spaces and can not be reduced to the metric setting in view of the remarks given in Karpinar et al. [2], thus our results fit with the nature of G-metric spaces. Moreover as some applications of best proximity point theorems, certain new fixed point results are established in the setting of G-metric spaces. Suitable examples are also presented which substantiate the genuineness of our investigations in this note.

AMS subject classification: 47H10, 54H25.
Keywords: G-metric spaces, best proximity point, best proximity point theorem, generalized proximal cyclic weak ϕ-contraction mapping, fixed point.

1. Introduction and Preliminaries

It is apparent that the fixed point theory is one of the fundamental theories in nonlinear analysis which has various applications in different branches of mathematics. In 1922, Banach contraction mapping principle [3] emerged and is the most acknowledged and vital result in fixed point theory. It states that each contraction in a complete metric space has a unique fixed point. This principle not only guarantees the existence and uniqueness of the fixed point but also demonstrates how to evaluate this point. By virtue of this fact, there is a great number of generalizations of Banach contraction mapping principle in the literature. ([4]-[7] and the references cited therein).

In 2003 Kirk et al. [8] turned the area of investigation of fixed point by introducing the notion of cyclic contraction mapping and proved some fixed point theorems for the operators in the class of cyclic contraction along with the cyclic version of the Banach contraction principle [3].

In 2005, Eldred, Kirk and Veeramani [9] proved the existence of a best proximity point for relatively nonexpansive mappings by using the notion of proximal normal structure. In 2006, Eldred and Veeramani [10] applied the notion of cyclic contraction and gave sufficient condition for the existence of a best proximity point for a cyclic contraction mapping T on a uniformly convex Banach space.

Fixed point theory is an important tool for solving the various equations of the form $Tx = x$ for self-mappings T defined on subsets of metric spaces. On the other hand, consider a non-self mapping T from A to B, where A and B are two nonempty subsets of a metric space. Since T is not a self-mapping, the equation $Tx = x$ is improbable to have a solution. In this case therefore, it is of key significance to search for an element x that is in some sense is neighboring to Tx. That is, when the equation $Tx = x$ has no solution, one tries to determine an approximate value of x subject to the condition that the distance between x and Tx is minimum.

Best approximation theorems and best proximity point theorems are relevant in this perspective. One of the most interesting theorem is due to Fan [11], called best approximation theorem. There have been many succeeding generalizations of Fan‘s Theorem (see [12]-[17] and references therein). Through best approximation theorems, we can guarantees the existence of approximate solutions but such solutions need not acquiesce the optimal solutions. But, best proximity point theorems provide sufficient conditions that assure the existence of approximate solutions which are optimal as well. Indeed, if there is no exact solution to the fixed point equation $Tx = x$ for a non-self mapping $T : A \to B$, then a best proximity theorem presents sufficient conditions for the existence of an optimal approximate solution x, called a best proximity point of the mapping T , satisfying the condition that $d(x, Tx) = d(A, B)$. A best proximity point theorem for non-self proximal contractions has been investigated in [18].

In the case of cyclic contractive mapping $T : A \cup B \to A \cup B$, a point $x \in A \cup B$ is called the best proximity point if $d(x, Tx) = d(A, B)$. Notice that a best proximity point x is a fixed point of T whenever $A \cap B \neq \emptyset$. Thus it generalizes the notion of fixed point in case when $A \cap B = \emptyset$. Further [19]-[24] examine several variants of contractions for the existence of a best proximity point.

On the other hand, in 2004, Mustafa and Sims ([25],[26]) introduced a new generalized metric space structure and called it, G-metric space. The authors also portrayed some fixed point theorems in perspective of G-metric spaces ([27]-[29]). Tagging on these initial papers, several researchers established many fixed point results on the setting of G-metric spaces (see [30]-[33]).

Consistent with Mustafa and Sims ([25],[26]), the following definitions and results will be needed in the sequel.

Definition 1.1. [26] Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G-1) $G(x, y, z) = 0$ if $x = y = z$;

(G-2) $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;

(G-3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(G-4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ symmetry in all three variables;

(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

The function G is called a generalized metric or a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2. [26] Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X. Then we say that the sequence $\{x_n\}$ is G-convergent to $x \in X$ if

$$\lim_{n,m \to +\infty} G(x, x_n, x_m) = 0,$$

that is, for any $\epsilon > 0$, there exists $N \in \mathcal{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $m, n > N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n,m \to +\infty} x_n = x$.

Proposition 1.3. [26] Let (X, G) be a G-metric space. Then the followings are equivalent:

(1) $\{x_n\}$ is G-convergent to x;

(2) $G(x_n, x_n, x) \to 0$ as $n \to +\infty$;

(3) $G(x_n, x, x) \to 0$ as $n \to +\infty$;

(4) $G(x_n, x_m, x) \to 0$ as $n, m \to +\infty$.

Definition 1.4. [26] Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathcal{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 1.5. [26] Let (X, G) be a G-metric space. Then the followings are equivalent:

(1) $\{x_n\}$ is a G-Cauchy sequence;

(2) For every $\epsilon > 0$, there is $N \in \mathcal{N}$ such that $G(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 1.6. [26] A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition 1.7. [26] Let (X, G) be a G-metric space. A mapping $F : X \times X \times X \to X$ is said to be continuous if, for any three G-convergent sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converging to x, y and z, respectively, $\{F(x_n, y_n, z_n)\}$ is G-convergent to $F(x, y, z)$.

Lemma 1.8. [26] By the rectangle inequality (G-5) together with the symmetry (G-4), we have

$$\begin{aligned} G(x, y, y) &= G(y, y, x) \\ &\leq G(y, x, x) + G(x, y, x) \\ &= 2G(y, x, x). \end{aligned}$$

Every G-metric on X defines a metric d_G on X given by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \qquad \text{for } all \ x, y \in X.$$

Recently, Hussain [1] introduced best proximity point concept in G-metric spaces and established the best proximity point theorems for new class of proximal contraction mappings.

Let (X, G) be a G-metric space. Suppose that A and B are nonempty subsets of a G-metric space (X, G). Then

$$\begin{aligned} A_0 &= \Big\{x \in A : d_G(x, y) = d_G(A, B) \ \ for \ \ some \ \ y \in B\Big\}, \\ B_0 &= \Big\{y \in B : d_G(x, y) = d_G(A, B) \ \ for \ \ some \ \ x \in A\Big\}, \end{aligned} \tag{1}$$

where $d_G(A, B) = \inf \Big\{d_G(x, y) : x \in A, y \in B\Big\}$.

If $T : A \cup B \to A \cup B$ is a cyclic contractive mapping in a G-metric space (X, G), a point $x \in A \cup B$ is called the best proximity point of T if $d_G(x, Tx) = d_G(A, B)$ or equivalently we can say $G(x, Tx, Tx) = G(A, B, B)$ and $G(x, x, Tx) = G(A, A, B)$, where

$$G(A, B, B) = \inf \Big\{G(a, b, b) : a \in A, b \in B\Big\}$$

and

$$G(A, A, B) = \inf \Big\{G(a, a, b) : a \in A, b \in B\Big\}.$$

Definition 1.9. [1] Let (X, G) be a G-metric space and let A and B be two nonempty subsets of X. Then B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ in B, satisfying the condition $d_G(x, y_n) \to d_G(x, B)$ for some x in A, has a G-convergent subsequence.

Very recently in ([34]), authors proved some proximity point theorems in G-metric spaces. In the sequel, we present some best proximity point theorems for more generalized conditions. Moreover as application, some new fixed point results are established in the framework of G-metric spaces.

2. Main Results

First of all generalized proximal cyclic weak ϕ contractive mapping is defined in the framework of G-metric spaces.

Definition 2.1. Let $\{A_i\}_{i=1}^{l}$ be a family of non-empty sub sets of G-metric space (X, G) such that $Y = \bigcup_{i=1}^{l} A_i$. Let $T : Y \to Y$ be a mapping satisfying $T(A_i) \subseteq A_{i+1}$, $i = 1, 2, \ldots, l$, where $A_{l+1} = A_1$. The mapping T is said to be generalized proximal cyclic weak ϕ-contractive, if for $x, u, u^* \in A_i$ and $y, v \in A_{i+1}$,

$$d_G(u^*, Tx) = d_G(A_i, A_{i+1}), \quad d_G(u, Tu^*) = d_G(A_i, A_{i+1}),$$

and

$$d_G(v, Ty) = d_G(A_i, A_{i+1}),$$

then we have

$$G(u^*, u, v) \leq M(x, v, y) - \phi(M(x, v, y)), \tag{2}$$

where $M(x, v, y) = \min\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$ and $\phi \in \Phi$, the set of continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$ and $i = 1, 2, \ldots, l$.

Example 2.2. For $X = R$, let $G : X \times X \times X \to R^+$ be defined by

$$G(x, y, z) = \frac{1}{2} \max\left\{ |x - y|, |y - z|, |z - x| \right\}.$$

Then (X, G) is a $G-$metric space and $d_G(x, y) = |x - y|$. For $A = \{1, 2, 3\}, B = \{0, 4, 5\}$, Let $T : A \cup B \to A \cup B$ be defined by $T(1) = 4, T(2) = 5, T(3) = 4, T(0) = 3, T(4) = T(5) = 1$. Then, clearly $T(A) \subseteq B$ and $T(B) \subseteq A, d_G(A, B) = 1$ and T is a generalized proximal cyclic weak ϕ-contraction for $u = 3, u^* = 3, x = 1 \in A$ and $v = 4, y = 0 \in B$. In fact, if

$$d_G(u^*, Tx) = d_G(A, B), \quad d_G(u, Tu^*) = d_G(A, B),$$

and

$$d_G(v, Ty) = d_G(A, B),$$

then we have

$$G(u^*, u, v) \leq M(x, v, y) - \phi(M(x, v, y)),$$

where $M(x, v, y) = \min\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$ and $\phi(t) = \frac{t}{2}$.

Following theorem is proved for generalized proximal cyclic weak ϕ-contractive mappings.

Theorem 2.3. Let (X, G) be a G-metric space and $\{A_i\}_{i=1}^m$ be a family of disjoint nonempty subsets of X with $Y = \bigcup_{i=1}^{l} A_i$ such that Y is a G-complete subspace of X. Let $T : Y \to Y$ be a generalized proximal cyclic weak ϕ-contractive mapping i.e., $T : Y \to Y$ be a mapping satisfying $T(A_i) \subseteq A_{i+1},\ i = 1, 2, \ldots, l$, where $A_{l+1} = A_1$ and if for $x, u, u^* \in A_i$ and $y, v \in A_{i+1}$,

$$d_G(u^*, Tx) = d_G(A_i, A_{i+1}),\ \ d_G(u, Tu^*) = d_G(A_i, A_{i+1}),$$

and

$$d_G(v, Ty) = d_G(A_i, A_{i+1}),$$

then

$$G(u^*, u, v) \le M(x, v, y) - \phi(M(x, v, y)), \tag{3}$$

where $M(x, v, y) = \min\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}, \phi \in \Phi$ and $i = 1, 2, \ldots, l$.
Suppose that A_{i_0} is nonempty such that $T(A_{i_0}) \subseteq A_{(i+1)_0}$ and A_{i+1} is approximatively compact with respect to A_i, for $i = 1, 2, \ldots, l$. Then T has a best proximity point in Y.

Proof. First of all, we construct a sequence of Picard iteration as usual. Define the sequence $\{x_n\}$ as $x_n = Tx_{n-1}$, n= 1,2,3,.... It is given that each A_{i_0} is nonempty then we pick an arbitrary point $x_0 \in A_{1_0} \subseteq A_1$. Since T is cyclic,

$$x_1 = Tx_o \in T(A_{1_0}) \subseteq A_{2_0} \subseteq A_2.$$

Then, we have

$$d_G(x_0, Tx_0) = d_G(A_1, A_2)$$

or

$$d_G(x_0, x_1) = d_G(A_1, A_2),$$

where $x_1 \in A_{2_0}$. Hence we must have

$$x_2 = Tx_1 \in T(A_{2_0}) \subseteq A_{3_0} \subseteq A_3,$$

then

$$d_G(x_1, Tx_1) = d_G(A_2, A_3)$$

or

$$d_G(x_1, x_2) = d_G(A_2, A_3).$$

Recursively, we obtain a sequence $\{x_n\}$ in $\bigcup_{i=1}^{l} A_{i_0} \subseteq \bigcup_{i=1}^{l} A_i$ satisfying

$$d_G(x_n, x_{n+1}) = d_G(A_i, A_{i+1})$$

for $n \in N$ and $i = 1, 2, 3, \ldots, l$. Also from above setting we conclude that $x_n \neq x_{n+1}$ since each A_i is disjoint, where $i = 1, 2, 3, \ldots, l$.

Now clearly

$$d_G(u^*, Tx) = d_G(A_i, A_{i+1}),$$
$$d_G(u, Tu^*) = d_G(A_i, A_{i+1}),$$
$$d_G(v, Ty) = d_G(A_i, A_{i+1}),$$

with $u^* = x_{n+1}, x = x_{n-1}, u = x_{n+1}, y = x_n, v = x_n$ and $i = 1, 2, 3, \ldots, l$. Therefore from (3), we obtain

$$G(x_{n+1}, x_{n+1}, x_n) \leq M(x_{n-1}, x_n, x_n) - \phi(M(x_{n-1}, x_n, x_n)), \tag{4}$$

where

$$\begin{aligned} M(x_{n-1}, x_n, x_n) &= \min\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, Tx_n, Tx_n), G(x_n, Tx_n, Tx_n)\} \\ &= \min\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \\ &= \min\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned}$$

Now, if

$$M(x_{n-1}, x_n, x_n) = G(x_n, x_{n+1}, x_{n+1}),$$

then from (4), we have

$$G(x_n, x_{n+1}, x_{n+1}) = G(x_n, x_{n+1}, x_{n+1}) - \phi(G(x_n, x_{n+1}, x_{n+1})),$$

and, then

$$\phi(G(x_n, x_{n+1}, x_{n+1})) = 0.$$

Hence we have

$$G(x_n, x_{n+1}, x_{n+1})) = 0,$$

it implies that $x_n = x_{n+1}$, which is a contradictione, in our setting. Then we have

$$M(x_{n-1}, x_n, x_n) = G(x_{n-1}, x_n, x_n)$$

Therefore

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &\leq G(x_{n-1}, x_n, x_n) - \phi(G(x_{n-1}, x_n, x_n)) \\ &\leq G(x_{n-1}, x_n, x_n). \end{aligned} \tag{5}$$

Thus sequence $G(x_n, x_{n+1}, x_{n+1})$ is a nonnegative, nonincreasing sequence which converges to $L \geq 0$. Letting $n \to \infty$ in (5), we obtain

$$L \leq L - \phi(L),$$

it implies that $\phi(L) = 0$, and so, $L = 0$, *i.e.*,

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{6}$$

Now we claim that $\{x_n\}$ is a G-Cauchy sequence in Y. On the contrary, we assume that $\{x_n\}$ is not G-Cauchy. Then there exist an $\epsilon > 0$ and corresponding subsequences $\{n(k)\}$ and $\{m(k)\}$ of N satisfying $n(k) > m(k) > k$ such that

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \epsilon, \tag{7}$$

where $n(k)$ is chosen as the smallest integer satisfying (7), i.e.,

$$G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) < \epsilon. \tag{8}$$

It is easy to conclude from (7), (8) and the rectangle inequality, that

$$\begin{aligned} \epsilon &\leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &\leq G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\ &< \epsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}). \end{aligned} \tag{9}$$

Taking limit $k \to \infty$ in (9) and utilizing (6), we obtain

$$\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon. \tag{10}$$

Observe that for every $k \in N$, there exist $s(k)$ satisfying $0 \leq s(k) \leq l$ such that

$$n(k) - m(k) + s(k) \equiv 1(l). \tag{11}$$

Therefore, for the sufficiently large values of k, we have $r(k) = m(k) - s(k) > 0$ and $x_{r(k)}$ and $x_{n(k)}$ lie in the consequative sets A_i and A_{i+1} respectively, where $0 \leq i \leq l$. Next using (3) with $u^* = x_{r(k)}$, $u = x_{n(k)+1}$, $v = x_{n(k)}$, $x = x_{r(k)}$, $y = x_{n(k)}$, we obtain

$$G(x_{r(k)}, x_{n(k)+1}, x_{n(k)}) \leq M(x_{r(k)}, x_{n(k)}, x_{n(k)}) - \phi(M(x_{r(k)}, x_{n(k)}, x_{n(k)})), \tag{12}$$

where

$$\begin{aligned} &M(x_{r(k)}, x_{n(k)}, x_{n(k)}) \\ &= \min\{G(x_{r(k)}, x_{n(k)}, x_{n(k)}), G(x_{r(k)}, Tx_{r(k)}, Tx_{r(k)}), G(x_{n(k)}, Tx_{n(k)}, Tx_{n(k)})\} \\ &= \min\{G(x_{r(k)}, x_{n(k)}, x_{n(k)}), G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}), G(x_{n(k)}, x_{n(k)+1}, Tx_{n(k)+1}.\} \end{aligned}$$

Employing rectangle inequality repeatedly, we observe that

$$\begin{aligned}
&G(x_{r(k)}, x_{n(k)}, x_{n(k)}) \\
&\leq G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}) + G(x_{r(k)+1}, x_{n(k)}, x_{n(k)}) \\
&\leq G(x_{r(k)}, x_{r(k)+1}, x_{r(k)+1}) + G(x_{r(k)+1}, x_{r(k)+2}, x_{r(k)+2}) + G(x_{r(k)+2}, x_{n(k)}, x_{n(k)}) \\
&\ \vdots \\
&\leq \sum_{i=r}^{m-1} [G(x_{i(k)}, x_{i(k)+1}, x_{i(k)+1})] + G(x_{m(k)}, x_{n(k)}, x_{n(k)})
\end{aligned}$$

or equivalently

$$0 \leq G(x_{r(k)}, x_{n(k)}, x_{n(k)}) - G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \leq \sum_{i=r}^{m-1} G(x_{i(k)}, x_{i(k)+1}, x_{i(k)+1}). \quad (13)$$

Note that the sum on the right hand side of (13) contains $s - 1 \leq l$ (finite) number of terms and due to (6) each term of this sum tends to 0 as $k \to \infty$. Therefore,

$$\lim_{n\to\infty} G(x_{r(k)}, x_{n(k)}, x_{n(k)}) = \lim_{n\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon. \quad (14)$$

Using rectangle inequality again, we have

$$\begin{aligned}
0 &\leq G(x_{r(k)}, x_{n(k)+1}, x_{n(k)}) \\
&\leq G(x_{r(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)}).
\end{aligned} \quad (15)$$

On letting $k \to \infty$ and using (14), we deduce that

$$\lim_{k\to\infty} G(x_{r(k)}, x_{n(k)+1}, x_{n(k)}) = \epsilon. \quad (16)$$

Now passing to limit as $k \to \infty$ in (12) and using (6), (14), (16), we get

$$\begin{aligned}
\epsilon &\leq \min\{\epsilon, 0, 0\} - \phi(\min\{\epsilon, 0, 0\}) \\
&\leq 0,
\end{aligned}$$

which contradicts the assumption that $\{x_n\}$ is not G-Cauchy. Thus $\{x_n\}$ is a G-Cauchy sequence in Y.

Since Y is complete, there exists $z \in Y$ such that $x_n \to z$ as $n \to \infty$. Without loss of generality, we assume that $z \in A_i$ for some i. On the other hand, for all $n \in N$, we can write, for each $i = 1, 2, \ldots, l$

$$\begin{aligned}
d_G(z, A_{i+1}) &\leq d_G(z, Tx_n) = d_G(z, x_{n+1}) \\
&\leq d_G(z, x_n) + d_G(x_n, x_{n+1}) \\
&\leq d_G(z, x_n) + d_G(A_i, A_{i+1}).
\end{aligned} \quad (17)$$

Taking limit as $n \to \infty$ in (17), we get

$$d_G(z, A_{i+1}) \le d_G(A_i, A_{i+1}),$$

but

$$d_G(A_i, A_{i+1}) \le d_G(z, A_{i+1}), \quad \text{for } z \in A_i.$$

So, we have

$$\lim_{n\to\infty} d_G(z, Tx_n) = d_G(z, A_{i+1}) = d_G(A_i, A_{i+1}). \tag{18}$$

Since A_{i+1} is approximatively compact with respect to A_i, the sequence $\{Tx_n\}$ has a subsequence $\{Tx_{n(K)}\}$ that converges to some $p \in A_{i+1}$. Hence

$$\begin{aligned} d_G(z, p) &= \lim_{n\to\infty} d_G(x_{n(k)}, Tx_{n(k)}) \\ &= \lim_{n\to\infty} d_G(x_{n(k)}, x_{n(k)+1}) \\ &= d_G(A_i, A_{i+1}) \end{aligned} \tag{19}$$

So $z \in A_{i_0}$. Now, since $T(z) \in T(A_{i_0}) \subseteq A_{(i+1)_0}$, there exists a $w \in A_{i_0}$ such that

$$d_G(w, Tz) = d_G(A_i, A_{i+1}).$$

Now we claim that $w = z$. For our assertion, utilizing (3) with $x = x_{n-1}$, $y = x_n$, $v = x_n$, $u = w$, $u^* = z$, we have

$$\begin{aligned} G(z, w, x_n) &\le M(x_{n-1}, x_n, x_n) - \phi(M(x_{n-1}, x_n, x_n)) \\ &\le \min\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} \\ &\quad - \phi(\min\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}). \end{aligned}$$

Letting $n \to \infty$, we have

$$G(z, z, w) \le G(z, z, z) - \phi(G(z, z, z)),$$

an so $G(z, z, w) = 0$, it implies that $z = w$. Hence we have,

$$d_G(z, Tz) = d_G(A_i, A_{i+1}). \tag{20}$$

Therefore T has the best proximity point in Y. ■

Following example substantiates the hypothesis of Theorem 2.3.

Example 2.4. Let $X = R$ and $G : X \times X \times X \to R^+$ be defined by

$$G(x, y, z) = \frac{1}{2}\max\{|x - y|, |y - z|, |z - x|\}.$$

Then clearly (X, G) is a G-metric space.

Now $d_G(x, y) = |x - y|$. Let $A = \{0, -1, -2, -3, -4\}$ and $B = \{1, 2, 3, 4\}$. Define $T : A \cup B \to A \cup B$ by

$$T(x) = \begin{cases} 1 & if\ x = -4, \\ 0 & if\ x = 4, \\ -x + 1 & otherwise. \end{cases}$$

Then clearly $T(A) \subseteq B$ and $T(B) \subseteq A$. Also taking $\phi(t) = \frac{t}{2}$. Clearly $d_G(A, B) = 1$ and $A_0 = \{0\}$. Now if we choose $u = 0, u^* = 0, x = -4 \in A$ and $v = 1, y = 4 \in B$, then

$$d_G(u^*, Tx) = d_G(A, B),$$

$$d_G(u, Tu^*) = d_G(A, B)$$

and

$$d_G(v, Ty) = d_G(A, B).$$

Now with $u = u^* = 0, x = -4, v = 1, y = 4$, we verify the Condition (3).

$$G(u^*, u, v) = G(0, 0, 1) = \frac{1}{2}.$$

Now

$$\begin{aligned} M(x, v, y) &= \min\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\} \\ &= \min\{G(-4, 1, 4), G(-4, 1, 1), G(4, 0, 0)\} \\ &= 2 \end{aligned}$$

and

$$M(x, v, y) - \phi(M(x, v, y)) = 2 - \frac{1}{2}(2) = 1.$$

Hence

$$G(u^*, u, v) = \frac{1}{2} \le 1 = M(x, v, y) - \phi(M(x, v, y)).$$

That is,

$$d_G(u^*, Tx) = d_G(A, B),$$

$$d_G(u, Tu^*) = d_G(A, B)$$

and

$$d_G(v, Ty) = d_G(A, B).$$

It implies that

$$G(u^*, u, v) \le M(x, v, y) - \phi(M(x, v, y)),$$

where

$$M(x, v, y) = \min\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}.$$

Thus T is a generalized weak ϕ-proximal cyclic mapping. All the conditions of Theorem 2.3 are satisfied and T has a best proximity point $z = 0 \in A \cup B$, since $d_G(0, T0) = d_G(A, B)$.

Now we obtain following corollaries (inspired by [33]).

Corollary 2.5. Let (X, G) be a G-metric space and $\{A_i\}_{i=1}^{m}$ be a family of disjoint non-empty subsets of X with $Y = \bigcup_{i=1}^{l} A_i$ such that Y is a G-complete subspace of X. Let $T : Y \to Y$ be a generalized proximal cyclic weak ϕ-contractive mapping, *i.e.*, $T : Y \to Y$ be a mapping satisfying $T(A_i) \subseteq A_{i+1},\ \ i = 1, 2, \dots, l$, where $A_{l+1} = A_1$ and if for $x, u, u^* \in A_i$ and $y, v \in A_{i+1}$,

$$d_G(u^*, Tx) = d_G(A_i, A_{i+1}),$$

$$d_G(u, Tu^*) = d_G(A_i, A_{i+1})$$

and

$$d_G(v, Ty) = d_G(A_i, A_{i+1}),$$

then

$$G(u^*, u, v) \le kM(x, v, y), \tag{21}$$

where $M(x, v, y) = \min\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}, \phi \in \Phi$ and $k \in (0, 1)$. Suppose that A_{i_0} is nonempty such that $T(A_{i_0}) \subseteq A_{(i+1)_0}$ and A_{i+1} is approximatively compact with respect to A_i, for $i = 1, 2, \dots, l$. Then T has a best proximity point in Y.

Proof. The proof is obvious by taking $\phi(t) = (1 - k)t, k \in (0, 1)$ in Theorem 2.3. ■

Corollary 2.6. Let (X, G) be a G-metric space and $\{A_i\}_{i=1}^{m}$ be a family of disjoint nonempty subsets of X with $Y = \bigcup_{i=1}^{l} A_i$ such that Y is a G-complete subspace of X. Let $T : Y \to Y$ be a generalized proximal cyclic weak ϕ-contractive mapping *i.e.* $T : Y \to Y$ be a mapping satisfying $T(A_i) \subseteq A_{i+1}$, for $i = 1, 2, \dots, l$, such that $A_{l+1} = A_1$ and if for $x, u, u^* \in A_i$ and $y, v \in A_{i+1}$,

$$d_G(u^*, Tx) = d_G(A_i, A_{i+1}),$$

$$d_G(u, Tu^*) = d_G(A_i, A_{i+1})$$

and

$$d_G(v, Ty) = d_G(A_i, A_{i+1}),$$

then

$$G(u^*, u, v) \le \alpha G(x, v, y) + \beta G(x, Tx, Tx) + \gamma G(y, Ty, Ty), \tag{22}$$

where $M(x, v, y) = \min\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}, \phi \in \Phi, \alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma < 1$. Suppose that A_{i_0} is nonempty such that $T(A_{i_0}) \subseteq A_{(i+1)_0}$ and A_{i+1} is approximatively compact with respect to A_i for $i = 1, 2, \ldots, l$. Then T has a best proximity point in Y.

Proof. Clearly, we have

$$\alpha G(x, v, y) + \beta G(x, Tx, Tx) + \gamma G(y, Ty, Ty) \leq (\alpha + \beta + \gamma)M(x, v, y).$$

Thus

$$G(u^*, u, v) \leq (\alpha + \beta + \gamma)M(x, v, y).$$

Using Corollary 2.5, with $k = (\alpha+\beta+\gamma) \in (0, 1)$, we obtain that T has a best proximity point. ■

Next corollary is obtained for cyclic mapping satisfying integral type contractive conditions.

Corollary 2.7. Let (X, G) be a G-metric space and $\{A_i\}_{i=1}^m$ be a family of disjoint nonempty subsets of X with $Y = \bigcup_{i=1}^{l} A_i$ such that Y is a G-complete subspace of X. Let $T : Y \to Y$ be a generalized proximal cyclic weak ϕ-contractive mapping, *i.e.*, $T : Y \to Y$ be a mapping satisfying $T(A_i) \subseteq A_{i+1}$, for $i = 1, 2, \ldots, l$ such that $A_{l+1} = A_1$ and if for $x, u, u^* \in A_i$ and $y, v \in A_{i+1}$,

$$d_G(u^*, Tx) = d_G(A_i, A_{i+1}),$$

$$d_G(u, Tu^*) = d_G(A_i, A_{i+1})$$

and

$$d_G(v, Ty) = d_G(A_i, A_{i+1}),$$

then

$$\int_o^{G(u*,u,v)} ds \leq \int_o^{M(x,v,y)} ds - \phi(\int_o^{M(x,v,y)} ds), \tag{23}$$

where $M(x, v, y) = \min\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$ and $\phi \in \Phi$. Suppose that A_{i_0} is nonempty such that $T(A_{i_0}) \subseteq A_{(i+1)_0}$ and A_{i+1} is approximatively compact with respect to A_i for $i = 1, 2, \ldots, l$. Then T has a best proximity point in Y.

Utilizing Corollary 2.5, we obtain next corollary.

Corollary 2.8. Let (X, G) be a G-metric space and $\{A_i\}_{i=1}^m$ be a family of disjoint nonempty subsets of X with $Y = \bigcup_{i=1}^{l} A_i$ such that Y is a G-complete subspace of X. Let $T : Y \to Y$ be a generalized proximal cyclic weak ϕ-contractive mapping *i.e.*, $T : Y \to Y$ be a mapping satisfying $T(A_i) \subseteq A_{i+1},\ i = 1, 2, \ldots, l$,such that $A_{l+1} = A_1$ and if for $x, u, u^* \in A_i$ and $y, v \in A_{i+1}$,

$$d_G(u^*, Tx) = d_G(A_i, A_{i+1}),$$

$$d_G(u, Tu^*) = d_G(A_i, A_{i+1})$$

and

$$d_G(v, Ty) = d_G(A_i, A_{i+1}),$$

then

$$\int_o^{G(u*,u,v)} ds \leq k \int_o^{M(x,v,y)} ds, \tag{24}$$

where $M(x, v, y) = \min\{G(x, v, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$ and $\phi \in \Phi$. Suppose that A_{i_0} is nonempty such that $T(A_{i_0}) \subseteq A_{(i+1)_0}$ and A_{i+1} is approximatively compact with respect to A_i for $i = 1, 2, \ldots, l$. Then T has a best proximity point in Y.

3. Application to fixed points theorems

We already discussed that in the case of cyclic contraction, a best proximity point is a fixed point of T whenever $\cap_{i=1}^{l} A_i \neq \phi$. Therefore utilizing the aforesaid concept to our best proximity results, we establish some new fixed point results for generalized cyclic contraction.

Let $X = Y = \cup_{i=1}^{l} A_i$. Then X is complete. Now, suppose each A_i is closed and $\cap_{i=1}^{l} A_i \neq \phi$, then certainly $d_G(A_i, A_{i+1}) = 0,\ \ i = 1, 2, \ldots, l$. Hence by

$$d_G(u^*, Tx) = d_G(A_i, A_{i+1}),$$

$$d_G(u, Tu^*) = d_G(A_i, A_{i+1})$$

and

$$d_G(v, Ty) = d_G(A_i, A_{i+1}),$$

it implies that $u^* = Tx$, $u = Tu^* = T^2x$ and $v = Ty$.

Now we are in a position to state our fixed point theorem.

Theorem 3.1. Let (X, G) be a complete G-metric space and $\{A_i\}_{i=1}^{l}$ be a family of non empty closed subsets of X with $X = \cup_{i=1}^{l} A_i$. Let $T : X \to X$ be a mapping satisfying $T(A_i) \subseteq T(A_{i+1})$ for $i = 1, 2, \ldots, l$ with $A_{i+1} = A_1$ and

$$G(Tx, T^2x, Ty) \leq M(x, Ty, y) - \phi(M(x, Ty, y)), \tag{25}$$

where $M(x, Ty, y) = \min\{G(x, Ty, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$, for all $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, l$ and $\phi \in \Phi$. Then T has a unique fixed point in $\cap_{i=1}^{l} A_i$.

Proof. Take an arbitrary $x_0 \in X$ and define the sequence $\{x_n\}$ as $x_n = Tx_{n-1},\ \ n = 1, 2, 3, \ldots$. Since T is a cyclic, for $x_0 \in A_1$, $x_1 = Tx_0 \in A_2$, $x_2 = Tx_1 \in A_3, \ldots$ and so on. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in N$, then obviously the fixed point of T is x_{n_0}.

Assume that $x_{n+1} \neq x_n$ for all $n \in N$. Now arguing similarly as in the proof of Theorem 2.3, we conclude that $\{x_n\}$ is a $G-$Cauchy sequence in X. Since $X = \cup_{i=1}^{l} A_i$ is $G-$complete, $\{x_n\}$ converges to a point $p \in X$.

Now it is easy to show that $p \in \cap_{i=1}^{l} A_i$. Since $x_0 \in A_1$, the subsequence $\{x_{l(n-1)}\}_{n=1}^{\infty} \subset A_1$, the subsequence $\{x_{l(n-1)+1}\}_{n=1}^{\infty} \subset A_2$ and continuing in this way, the subsequence $\{x_{ln-1}\}_{n=1}^{\infty} \subset A_l$. Since all the l-subsequences are G-convergent in closed sets A_i and therefore they converge to the same limit $p \in \cap_{i=1}^{l} A_i$.

Finally we show that the limit p is the fixed point of T, i.e., $p = Tp$. Now from (25) with $x = x_n$ and $y = p$ (knowing that $x_n = Tx_{n-1}$),

$$G(Tx_n, T^2x_n, Tp) \leq M(x_n, Tp, p) - \phi(M(x_n, Tp, p)),$$

where $M(x_n, Tp, p) = \min\{G(x_n, Tp, p), G(x_n, Tx_n, Tx_n), G(p, Tp, Tp)\}$. Taking $n \to \infty$, we get

$$G(p, p, Tp) \leq 0 - \phi(0),$$

and so, $G(p, p, Tp) = 0$, it implies that $Tp = p$. Therefore p is fixed point of T.

Next we prove the uniqueness of p. Assume that $q \in X$ is another fixed point of T. Now putting $x = p$ and $y = q$ in (25), this yields

$$G(Tp, T^2p, Tq) \leq M(p, Tq, q) - \phi(M(p, Tq, q)),$$

where $M(p, Tq, q) = \min\{G(p, Tq, q), G(p, Tp, Tp), G(q, Tq, Tq)\} = 0$. Hence we have $G(p, p, q) \leq 0 - \phi(0)$, and so $G(p, p, q) = 0$, it implies that $p = q$. ■

Applying same argument as in previous section, we have the following corollaries:

Corollary 3.2. Let (X, G) be a complete G-metric space and $\{A_i\}_{i-1}^{l}$ be a family of nonempty closed subsets of X with $X = \cup_{i=1}^{l} A_i$. Let $T : X \to X$ be a map satisfying $T(A_i) \subseteq T(A_{i+1})$, for $i = 1, 2, \ldots, l$ with $A_{i+1} = A_1$ and

$$G(Tx, T^2x, Ty) \leq kM(x, Ty, y), \tag{26}$$

where $M(x, Ty, y) = \min\{G(x, Ty, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$, $k \in (0, 1)$ for all $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, l$ and $\phi \in \Phi$. Then T has a unique fixed point in $\cap_{i=1}^{l} A_i$.

Corollary 3.3. Let (X, G) be a complete G-metric space and $\{A_i\}_{i-1}^{l}$ be a family of nonempty closed subsets of X with $X = \cup_{i=1}^{l} A_i$. Let $T : X \to X$ be a map satisfying $T(A_i) \subseteq T(A_{i+1})$, for $i = 1, 2, \ldots, l$ with $A_{i+1} = A_1$ and

$$G(Tx, T^2x, Ty) \leq \alpha G(x, Ty, y) + \beta G(x, Tx, Tx) + \gamma G(y, Ty, Ty), \tag{27}$$

where $M(x, Ty, y) = \min\{G(x, Ty, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$ for $x \in A_i$, $y \in A_{i+1}$, $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma > 1$, and $\phi \in \Phi$. Then T has a unique fixed point in $\cap_{i=1}^{l} A_i$.

Corollary 3.4. Let (X, G) be a complete G-metric space and $\{A_i\}_{i-1}^{l}$ be a family of nonempty closed subsets of X with $X = \cup_{i=1}^{l} A_i$. Let $T : X \to X$ be a map satisfying $T(A_i) \subseteq T(A_{i+1})$ for $i = 1, 2, \ldots, l$ with $A_{i+1} = A_1$ and

$$\int_0^{G(Tx,T^2x,Ty)} ds \leq \int_0^{M(x,Ty,y)} ds - \phi(\int_0^{M(x,Ty,y)} ds), \tag{28}$$

where $M(x, Ty, y) = \min\{G(x, Ty, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$, for $x \in A_i$, $y \in A_{i+1}$, and $\phi \in \Phi$. Then T has a unique fixed point in $\cap_{i=1}^{l} A_i$.

Corollary 3.5. Let (X, G) be a complete G-metric space and $\{A_i\}_{i-1}^{l}$ be a family of nonempty closed subsets of X with $X = \cup_{i=1}^{l} A_i$. Let $T : X \to X$ be a map satisfying $T(A_i) \subseteq T(A_{i+1})$, for $i = 1, 2, \ldots, l$ with $A_{i+1} = A_1$ and

$$\int_0^{G(Tx,T^2x,Ty)} ds \leq k \int_0^{M(x,Ty,y)} ds, \tag{29}$$

where $k \in (0, 1)$, $x \in A_i$, $y \in A_{i+1}$, and $\phi \in \Phi$. Then T has a unique fixed point in $\cap_{i=1}^{l} A_i$.

Next some examples to demonstrate the validity of the hypothesis of Theorem 3.1, are furnished.

Example 3.6. Let $X = A_1 \cup A_2$, where $A_1 = \{(a, 0) : 0 \leq a \leq 1\}$ and $A_2 = \{(0, b) : b = \frac{a}{2}$ and $0 \leq a \leq 1\}$. Let $G : X^3 \to [0, \infty)$ be defined by

$$G((x_1, y_1), (x_2, y_2), (x_3, y_3)) = \sum_{i=1}^{3} [|x_i - x_{i+1}| + |y_i - y_{i+1}|],$$

where $(x_4, y_4) = (x_1, y_1)$. Then (X, G) is a complete $G-$metric space. Clearly A_1 and A_2 are G-closed subsets of X.

Consider the mapping $T : A_1 \cup A_2 \to A_1 \cup A_2$ given by

$$T(a, 0) = (0, \frac{a}{4}) \text{ and } T(0, b) = (\frac{b}{4}, 0).$$

Clearly $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$.

Next to verify that T satisfies the contraction condition (25), with $\phi(t) = \frac{t}{4}$. Consider $(x, y) \in A_1 \times A_2$ and let $x = (a, 0)$, $y = (0, b)$ $\left(\text{knowing that } b = \frac{a}{2}, 0 \leq a \leq 1\right)$. Then $Tx = \left(0, \frac{a}{4}\right)$ and $Ty = \left(\frac{b}{4}, 0\right)$. Now we observe that

(1) $G(Tx, T^2x, Ty) = G\left(\left(0, \frac{a}{4}\right), \left(\frac{a}{16}, 0\right), \left(\frac{b}{4}, 0\right)\right) = \frac{3}{4}a,$

(2) $G(x, Ty, y) = G\left((a, 0), \left(\frac{b}{4}, 0\right), (0, b)\right) = 3a,$

(3) $G(x, Tx, Tx) = G\left((a, 0), \left(0, \frac{a}{4}\right), \left(0, \frac{a}{4}\right)\right) = \frac{5}{2}a,$

(4) $G(y, Ty, Ty) = G\left((0, b), \left(\frac{b}{4}, 0\right), \left(\frac{b}{4}, 0\right)\right) = \frac{5}{2}b = \frac{5}{4}a,$

and then

$$M(x, Ty, y) = \min\{G(x, Ty, y), G(x, Tx, x), G(y, Ty, y)\} = \frac{5}{4}a$$

and

$$\phi(M(x, Ty, y)) = \frac{1}{4}\left(\frac{5}{4}a\right) = \frac{5}{16}a.$$

Hence, we have

$$M(x, Ty, y) - \phi(M(x, Ty, y)) = \frac{5}{4}a - \frac{5}{16}a = \frac{15}{16}a.$$

Thus, clearly $G(Tx, T^2x, Ty) \leq M(x, Ty, y) - \phi(M(x, Ty, y))$. Therefore, all the conditions of Theorem 3.1 are satisfied and the point $(0, 0) \in A_1 \cap A_2$ remains fixed under T and is indeed unique.

Example 3.7. Define a G-metric G on the set $X = \{a, b, c\}$ by

$$G(a, a, a) = G(b, b, b) = G(c, c, c) = 0,$$

$$G(a, a, b) = 4,$$

$$G(a, a, c) = G(a, b, b) = G(b, b, c) = 6$$

and

$$G(a, b, c) = G(a, c, c) = G(b, c, c) = 8$$

with symmetry in all variables. Since $G(x, y, y) \neq G(x, x, y)$, for $x \neq y$, (X, d) is not symmetric. Let $A_1 = \{a, b\}$ and $A_2 = \{b, c\}$. Consider the mapping $T : X \to X$ given by $Ta = b, Tb = b, Tc = b$. Undoubtedly $T : A_1 \cup A_2 \to A_1 \cup A_2$ is a cyclic mapping and $A_1 \cup A_2 = X$.

Now the following Table 3.1 shows that the contractive condition (25) is satisfied whenever $x \in A_1, y \in A_2$ or $y \in A_1, x \in A_2$ with $\phi(t) = t/6$.

(x, y)	$G(Tx, T^2x, Ty)$	$P = M(x, Ty, y)$	$Q = \phi(M(x, Ty, y))$	$P - Q$
(a, b)	0	0	0	0
(a, c)	0	6	1	5
(b, b)	0	0	0	0
(b, c)	0	0	0	0
(c, a)	0	6	1	5
(c, b)	0	0	0	0
(b, a)	0	0	0	0

Table 3.1

From the second and fifth columns, we conclude that

$$G(Tx, T^2x, Ty) \leq M(x, Ty, y) - \phi(M(x, Ty, y)),$$

where $M(x, Ty, y) = \min\{G(x, Ty, y), G(x, Tx, Tx), G(y, Ty, Ty)\}$. Therefore, all the conditions of Theorem 3.1 are fulfilled and it follows that T has a unique fixed point $b \in A_1 \cap A_2$.

Concluding Remark

A G-metric naturally induces a metric d_G given by

$$d_G(x, y) = G(x, y, y) + G(x, x, y).$$

Due to condition that x is not equal to y, either of the inequalities (3.1), (3.2), (3.3), (3.4) and (3.5) do not reduce to any metric inequality with the metric d_G. Thus our fixed point results are not the consequences any corresponding results on metric spaces from existing literature.

Acknowledgments

This work was supported by the Basic Science Research Program through the National Research Foundation (NRF) Grant funded by Ministry of Education of the republic of Korea (2014046293).

References

[1] Hussain N, Latif A, Salami P, 2014, "Best proximity point results in G- metric spaces", Abstract and applied Analysis, 2014, article ID 837943.

[2] Karpinar, E, Agrawal, R, 2013, "Further fixed point result on G-metric spaces", Fixed point theory and application, 2013:154.

[3] Banach, S., 1922, "Sur les operations dans les ensembles abstraits et leurs applications aux equations integrales", Fundam. Math. 3, 133–181.

[4] Arvanitakis, A. D., 2003, "A proof of the generalized Banach contraction conjecture", Proc. Am. Math. Soc. 131(12), 3647–3656.

[5] Boyd, D W, Wong, J S W, 1969, "On nonlinear contractions", Proc. Am. Math. Soc. 20, 458–464.

[6] Merryfield, J, Rothschild, B, Stein, JD Jr, 2002, "An application of Ramsey's theorem to the Banach contraction principle", Proc. Am. Math. Soc. 130(4), 927–933.

[7] Suzuki, T, 2008, "A generalized Banach contraction principle that characterizes metric completeness", Proc. Am. Math. Soc. 136(5), 1861–1869.

[8] Kirk, W A, Srinavasan, P S and Veeramani, P, 2003, "Fixed points for mapping satisfying cyclical contractive conditions", Fixed point theory, 4 79–89.

[9] Eldred, A A, Kirk, W A, Veeramani, P , 2005, "Proximinal normal structure and relatively non expanisve mappings", Stud. Math. 171(3), 283–93.

[10] Eldred, A A, Veeramani, P, 2006, "Existence and convergence of best proximity points", J. Math. Anal. Appl., 323, 1001–006.

[11] Fan, K, 1969, "Extensions of two fixed point theorems of F. E. Browder", Math. Z. 112, 234–40.

[12] Prolla, J B, 1982–83, "Fixed point theorems for set valued mappings and existence of best approximations", Numer. Funct. Anal. Optim. 5, 449–455.

[13] Reich, S, 1978, "Approximate selections, best approximations, fixed points and invariant sets", J. Math. Anal. Appl. 62, 104–113.

[14] Sehgal, V M, Singh, S P, 1988, "A generalization to multi functions of Fans best approximation theorem", Proc. Am. Math. Soc. 102, 534–37.

[15] Sehgal, V M, Singh, S P, 1989, "A theorem on best approximations", Numer. Funct. Anal. Optim. 10, 181–184.

[16] Srinivasan, P S, 2011, "Best proximity pair theorems", Acta Sci. Math. 67, 421–429.

[17] Vetrivel, V, Veeramani, P, Bhattacharyya, P, 1992, "Some extensions of Fans best approximation theorem", Numer. Funct. Anal. Optim. 13, 397–402.

[18] Sadiq Basha, S, 2011, "Best proximity points: global optimal approximate solutions", J. Glob. Optim. 49.

[19] Al-Thagafi, M A, Shahzad, N, 2009, "Convergence and existence results for best proximity points", Non-linear Anal. 70(10), 3665–671.

[20] Di Bari, C, Suzuki, T, Vetro, C, 2008, "Best proximity points for cyclic Meir-Keeler contractions", Non-linear Anal. 69(11), 3790–3794.

[21] Karpagam, S, Agrawal, S, 2009, "Best proximity point theorems for p-cyclic Meir-Keeler contractions", Fixed Point Theory Appl., Art. ID 197308.

[22] Hussain, N, Kutbi M A and Salimi P, 2013, "Best proximity point results for modified $\alpha - \psi$ proximal rational contractions", Abstract and Applied Analysis, vol. 2013, Article ID 927457, 14 pages.

[23] Hussain N, Latif, A and Salimi P, 2014, "Best proximity point results for modified Suzuki $\alpha - \psi$ proximal contractions", Fixed Point Theory and Applications, vol. 2014, article 10.

[24] Anuradha, J, Veeramani, P, 2009, "Proximal point-wise contraction", Topol. Appl. 156(18), 2942–948.

[25] Mustafa, Z and Sims, B, 2004, "Some remarks concerning D-metric spaces", Proceedings of International Conference on Fixed Point Theory and Applications, Yokohama Publishers, Valencia Spain, July 13–19(2004), 189–198.

[26] Mustafa, Z and Sims, B, 2006, "A new approach to a generalized metric spaces", J. Nonlinear Convex Anal., 7, 289–297.

[27] Mustafa, Z and Sims, B, 2009, "Fixed point theorems for contractive mappings in complete G-metric spaces", Fixed Point Theory and Applications, (2009) Article ID 917175, 10 pages.

[28] Mustafa, Z, Obiedat, H, Awawdeh, F, 2008, "Some fixed point theorems for mappings on complete G-metric spaces", Fixed point theory and Applications, Article ID 18970, 12 pages.

[29] Abbas M, Kim J K and Nazir T, 2015, "Common fixed point of mappings satisfying almost generalized contractive condition in partially ordered G-metric spaces", J. of Comput. Anal. and Appl., 19, 928–937.

[30] Jleli, M, Samet, B, 2012, "Remarks on G-metric Spaces and fixed point theorem", Fixed point theory and application, 2012, article ID 210.

[31] Hussain, N, Karapinar, E, Salimi, P and Vetro, P, 2013, "Fixed point results for G^m-Meir-Keeler contractive and $G - (\alpha, \psi)$-Meir-Keeler contractive mappings", Fixed Point Theory and Applications, vol. 2013, article 34.

[32] Salimi, P and Vetro, P, 2014, "A result of Suzuki type in partial G-metric spaces", Acta Mathematica Scientia B, vol. 34, no. 2, article 111.

[33] Karpinar, E, Yildiz-Ulas, A and Erhan, I M 2012, "Cyclic contraction on G-metric spaces", Abstract and Applied Analysis, volume 2012, Art. ID-182947.

[34] Singh, D, Joshi, V, 2014, "Some best proximity point theorems in G-metric spaces", Presented in international Conference on Geometric Functional Analysis and its applications, 27 Oct. 31Oct., 2014 at Besancon, France.