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Blow up solutions for a general Gierer-Meinhardt system with m components

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Abstract

This article deals with a general Gierer-Meinhardt model contains an activator and $(m-1)$ inhibitors, formed Reaction-Diffusion system. The paper aims at proving the existence of space-independent initial data such that the solutions of this system blow up in finite time.

AMS subject classification:
Keywords: Reaction-Diffusion Systems, Gierer-Meinhardt model, Blow up solutions, Activator, Inhibitor.

1. Introduction

In 1972, A. Gierer and H. Meinhardt proposed molecular models of reaction-diffusion type describe the pattern formation in Biological environment [1], among there models the activator-inhibitor model. This paper deals the study of existence of blow up solutions without diffusion of a general model contains an activator with concentration $(u_1)$and $(m-1)$ inhibitors with concentrations$\left(u_j\right)$, $j = 2, \ldots, m$.

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = a_1 \Delta u_1 - b_1 u_1 + \dfrac{u_1^{p_{11}}}{\prod\limits_{i=2}^{m} u_i^{p_{1i}}} + \sigma \\ \dfrac{\partial u_j}{\partial t} = a_j \Delta u_j - b_j u_j + \dfrac{u_1^{p_{j1}}}{\prod\limits_{i=2}^{m} u_i^{p_{ji}}}, \qquad j = 2, \ldots, m \end{cases} , x \in \Omega \, , t > 0 \tag{1.1}$$

with Neumann boundary conditions

$$\frac{\partial u_j}{\partial \eta} = 0, \quad x \in \partial\Omega, t > 0, \quad \text{for } j = 1, \ldots, m \tag{1.2}$$

and the initial data

$$u_j(x, 0) = \varphi_j(x) > 0, \quad x \in \Omega, \quad \text{for all } j = 1, \ldots, m \tag{1.3}$$

here Ω is an open bounded domain of class $\mathbb{C}^1$ in $\mathbb{R}^N$, with boundary $\partial\Omega$ and $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$.

Suppose that: $\sigma > 0$. a_j, b_j, p_{ji}: are non negative indexes for all: $i, j = 1, \ldots, m$ with

$$\frac{p_{11} - 1}{\max\limits_{j=2,m} \left(p_{j1}\right)} > \min\left[\frac{\prod\limits_{i=2}^{m} p_{1i}}{\min\limits_{j,i=2,m} \left(p_{ji}\right) - \max\limits_{i=2,m} \left(p_{1i}\right) + \prod\limits_{i=2}^{m} p_{1i}}, 1\right]. \tag{1.4}$$

Blow up occurs if there exist a time $T_{\max} < \infty$, such that the solutions are well defined for all $0 < t < T_{\max}$, while

$$\lim_{t \nearrow T_{\max}} \sum_{i=1}^{m} \|u_i(t, .)\|_{\infty} = \infty.$$

Li, Chen and Qin [5] proved the existence of blow up solutions for the activator-inhibitor model which is formed with two equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u - \mu u + u^p v^{-q} + \sigma \\ \dfrac{\partial v}{\partial t} = D\Delta v - \nu v + u^r v^{-s}, \end{cases}$$

for some initial values if either $r > p - 1$ with $rq < (p-1)(s+1)$ or $r < p - 1$. Quittner and Souplet [6] got the same results.

In the case of three components (an activator and two inhibitors)

$$\begin{cases} \dfrac{\partial u}{\partial t} = a_1 \Delta u - b_1 u + \dfrac{u^{p_1}}{v^{q_1} w^{r_1}} + \sigma \\ \dfrac{\partial v}{\partial t} = a_2 \Delta v - b_2 v + \dfrac{u^{p_2}}{v^{q_2} w^{r_2}} & x \in \Omega, t > 0, \\ \dfrac{\partial w}{\partial t} = a_3 \Delta w - b_3 w + \dfrac{u^{p_3}}{v^{q_3} w^{r_3}} \end{cases}$$

we proved the existence of space-independent initial data [3], such that the solutions blow up in finite time if

$$\frac{p_1 - 1}{\max(p_2, p_3)} > \min\left[\frac{q_1 r_1}{\min(q_2, q_3, r_2, r_3) - \max(q_1, r_1) + q_1 r_1}, 1\right].$$

2. Blow up results

Starting with a preliminary lemma which its proof is followed immediately from the maximum principle.

Lemma 2.1. Let $(u_1(t,.),\dots,u_m(t,.))$ be a solution of (1.1). Then for any (t,x) in $(0,T_{\max})\times\Omega$, we have

$$u_i\,(t,x)\geq e^{-b_it}\min\left(\varphi_i(x)\right)>0,\quad i=1,\dots,m \tag{2.1}$$

Theorem 2.2. Assume the condition (1.4), then there exist space-independent initial data $\varphi_i,\quad i=1,\dots,m$, such that the solution $(u_1,\dots,u_m)=(u_1(t),\dots,u_m(t))$ of problem (1.1) satisfies $T_{\max}<\infty$.

Proof. We consider space-independent solutions of (1.1), i.e. solutions of the corresponding ODE system without diffusion. For spatially homogeneous initial data $\varphi_i\geq 1,\ i=1,\dots,m$, we assume that $T_{\max}\left(\varphi_1,\dots,\varphi_m\right)>1$ (for find a contradiction). In what follows, all the positive constants $C_1,\ C_2,\dots$ are independent of $\varphi_i,\ i=1,\dots,m$.

For fixed $\alpha_i>0,\ i=1,\dots,m$, let $\lambda=\alpha_1b_1-\sum\limits_{i=2}^{m}\alpha_ib_i$ and $L\,(t)=\dfrac{u_1^{\alpha_1}}{\prod\limits_{i=2}^{m}u_i^{\alpha_i}}$.

We have

$$L'(t)=\alpha_1\frac{u_1^{\alpha_1-1}}{\prod\limits_{i=2}^{m}u_i^{\alpha_i}}\left(\frac{\partial u_1}{\partial t}\right)-\sum_{i=2}^{m}\alpha_i\frac{u_1^{\alpha_1}}{u_i^{\alpha_i+1}\prod\limits_{\substack{j=2\\ j\neq i}}^{m}u_j^{\alpha_j}}\left(\frac{\partial u_i}{\partial t}\right),$$

using equations of (1.1) without diffusion, we obtain

$$L'(t)=\left(\begin{array}{c}-b_1\alpha_1\dfrac{u_1^{\alpha_1}}{\prod\limits_{i=2}^{m}u_i^{\alpha i}}+\sum\limits_{i=2}^{m}b_i\alpha_i\dfrac{u_1^{\alpha_1}}{\prod\limits_{i=2}^{m}u_i^{\alpha i}}\\ +\alpha_1\dfrac{u_1^{p_{11}+\alpha_1-1}}{\prod\limits_{i=2}^{m}u_i^{p_{1i}+\alpha_i}}-\sum\limits_{j=2}^{m}\alpha_j\dfrac{u_1^{p_{j1}+\alpha_1}}{u_j^{\alpha_j+p_{jj}+1}\prod\limits_{\substack{i=2\\ i\neq j}}^{m}u_i^{p_{ji}+\alpha_i}}\\ +\sigma\alpha_1\dfrac{u_1^{\alpha_1-1}}{\prod\limits_{i=2}^{m}u_i^{\alpha_i}}\end{array}\right),$$

then we get

$$L'(t) + \lambda L(t) = \alpha_1 \frac{u_1^{p_{11}+\alpha_1-1}}{\prod\limits_{i=2}^{m} u_i^{p_{1i}+\alpha_i}} - \sum_{j=2}^{m} \alpha_j \frac{u_1^{p_{j1}+\alpha_1}}{u_j^{\alpha_j+p_{jj}+1} \prod\limits_{\substack{i=2\\ i\neq j}}^{m} u_i^{p_{ji}+\alpha_i}} + \sigma \alpha_1 \frac{u_1^{\alpha_1-1}}{\prod\limits_{i=2}^{m} u_i^{\alpha_i}} \qquad (2.2)$$

We consider two cases separately.

Case 1. $p_{11} - 1 > \max\limits_{j=\overline{2,m}} \left(p_{j1}\right)$. We apply (2.2) with $\alpha_1 = 1$. Taking α_i large enough,$i = \overline{2,m}$, with using (2.1), $\varphi_j(x) \geq 1$, $j = \overline{2,m}$, and Young's Inequality we have for all $t \in [0, 1]$

$$\frac{\alpha_1}{m} \frac{u_1^{p_{11}}}{\prod\limits_{i=2}^{m} u_i^{p_{1i}+\alpha_i}} = \frac{\alpha_1}{m} \left(\frac{u_1^{p_{j1}+1}}{\prod\limits_{i=2}^{m} u_i^{p_{ji}+1+\alpha_i}} \right)^{p_{11}/p_{j1}+1} \prod_{i=2}^{m} u_i^{h_{ij}} \geq \alpha_j \frac{u_1^{p_{j1}+\alpha_1}}{u_j^{\alpha_j+p_{jj}+1} \prod\limits_{\substack{i=2\\ i\neq j}}^{m} u_i^{p_{ji}+\alpha_i}} - C_j, \quad j = \overline{2,m}$$

where

$$h_{ij} = \left(p_{ji} + 1 + \alpha_j\right) \frac{p_{11}}{p_{j1}+1} - q_{1j} - \alpha_j > 0, \quad j = \overline{2,m}, \quad i = \overline{2,m}$$

$$C_j = \left(\frac{\alpha_1}{m}\right)^{1-\frac{p_{11}}{p_{11}-\left(p_{j1}+1\right)}} \alpha_j^{\frac{p_{11}}{p_{11}-\left(p_{j1}+1\right)}}, \quad j = \overline{2,m}.$$

For other hand with using (2.1) we have

$$\frac{\alpha_1}{m} \frac{u_1^{p_{11}}}{\prod\limits_{i=2}^{m} u_i^{p_{1i}+\alpha_i}} = \frac{\alpha_1}{m} \left(\frac{u_1}{\prod\limits_{i=2}^{m} u_i^{\alpha_i}} \right)^{p_{11}} \prod_{i=2}^{m} u_i^{d_i} \geq C_{m+1} \left(\frac{u_1}{\prod\limits_{i=2}^{m} u_i^{\alpha_i}} \right)^{p_{11}}$$

where

$$d_i = (p_{11} - 1)\,\alpha_i - p_{1i} > 0, \quad i = \overline{2,m}.$$

Then we get

$$\alpha_1 \frac{u_1^{p_{11}}}{\prod\limits_{i=2}^{m} u_i^{p_{1i}+\alpha_i}} \geq \sum_{j=2}^{m} \alpha_j \frac{u_1^{p_{j1}+\alpha_1}}{u_j^{\alpha_j+p_{jj}+1} \prod\limits_{\substack{i=2\\ i\neq j}}^{m} u_i^{p_{ji}+\alpha_i}} + C_{m+1} \left(\frac{u_1}{\prod\limits_{i=2}^{m} u_i^{\alpha_i}} \right)^{p_{11}} - \sum_{j=2}^{m} C_j$$

from (2.2), we have

$$L'(t)+\lambda L(t)\geq C_{m+1}\left(\frac{u_1}{\prod\limits_{i=2}^{m}u_i^{\alpha_i}}\right)^{p_{11}}-\sum_{j=2}^{m}C_j$$

or

$$L'(t)+\lambda L(t)\geq C_{m+1}\left(L(t)\right)^{p_{11}}-C$$

where $C=\sum\limits_{j=2}^{m}C_j$.

Taking $L(0)$ large enough, this implies blow up of u_1 before $t=1$; which contradict with the supposition that $T_{\max}\left(\varphi_1,\ldots,\varphi_m\right)>1$.

Case 2.

$$p_{11}-1<\max_{j=\overline{2,m}}\left(p_{j1}\right),$$

$$(p_{11}-1)\left[\min_{j,i=\overline{2,m}}\left(p_{ji}\right)-\max_{i=\overline{2,m}}\left(p_{1i}\right)+\prod_{i=2}^{m}p_{1i}\right]>\prod_{i=2}^{m}p_{1i}\max_{j=\overline{2,m}}\left(p_{j1}\right).$$

We claim that there exist constants $C_{m+2},C_{m+3}>0$ such that, if

$$\varphi_1^{\left(\max\limits_{j=\overline{2,m}}(p_{j1})-p_{11}+1\right)}\geq C_{m+2}\prod_{j=2}^{m}\varphi_j^{\left(\min\limits_{i=\overline{2,m}}(p_{ij})-q_{1j}\right)}\tag{2.3}$$

then

$$u_1^{\left(\max\limits_{j=\overline{2,m}}(p_{j1})-p_{11}+1\right)}\geq C_{m+3}\prod_{j=2}^{m}u_j^{\left(\min\limits_{i=\overline{2,m}}(p_{ij})-q_{1j}\right)},\ \text{ for }\ 0<t\leq 1.\tag{2.4}$$

To prove this, letting $\phi\left(t\right)=e^{\lambda t}L\left(t\right)$ and applying (2.2) with
$\alpha_1=\max\limits_{j=\overline{2,m}}\left(p_{j1}\right)-p_{11}+1>0,$
$\alpha_j=\min\limits_{i=\overline{2,m}}\left(p_{ij}\right)-q_{1j}\ ,\quad j=\overline{2,m}$
we see that, for all $t\in[0,1]$,

$$\phi'\left(t\right)\leq 0\Longrightarrow L'\left(t\right)+\lambda L\left(t\right)\leq 0,$$

which implies

$$\frac{u_1^{\max\limits_{j=\overline{2,m}}(p_{j1})+\alpha_1}}{\prod\limits_{j=2}^{m}u_j^{\left(\min\limits_{i=\overline{2,m}}(p_{ij})+\alpha_j\right)}}\geq\frac{u_1^{p_{11}-1+\alpha_1}}{\prod\limits_{j=2}^{m}u_j^{(p_{1j})+\alpha_j}}\alpha_1\left(\sum_{j=2}^{m}\alpha_j\right)^{-1},$$

then we get

$$\phi(t) \geq e^{-|\lambda|} \frac{u_1^{\alpha_1}}{\prod_{j=2}^{m} u_j^{\alpha_j}} \geq e^{-|\lambda|}\alpha_1 \left(\sum_{j=2}^{m} \alpha_j\right)^{-1} =: C_{m+2}.$$

Consequently we have $\phi(t) \geq C_{m+2}$ on $[0, 1]$, and the claim follows with

$$C_{m+3} = e^{|\lambda|} C_{m+2}.$$

Now assume (2.3). Using the first equation in (1.1) and (2.4), we deduce that

$$\begin{aligned} u_1' + b_1 u_1 &\geq \frac{u_1^{p_{11}}}{\prod_{j=2}^{m} u_j^{(p_{1j})}} \geq C_{m+4} u_1^{\left(p_{11} - \prod_{j=2}^{m} p_{1j} \frac{\max_{j=\overline{2,m}}(p_{j1}) - p_{11} + 1}{\min_{j,i=\overline{2,m}}(p_{ji}) - \max_{j=\overline{2,m}}(p_{1j})}\right)} \\ &= C_{m+4} u_1^{\theta}, \quad 0 < t \leq 1 \end{aligned}$$

where

$$C_{m+4} = C_{m+3}^{\left(\frac{\prod_{j=2}^{m} p_{1j}}{\min_{j,i=\overline{2,m}}(p_{ji}) - \max_{j=\overline{2,m}}(p_{1j})}\right)}$$

$$\begin{aligned} \theta &= p_{11} - \prod_{j=2}^{m} p_{1j} \frac{\max_{j=\overline{2,m}}(p_{j1}) - p_{11} + 1}{\min_{j,i=\overline{2,m}}(p_{ji}) - \max_{j=\overline{2,m}}(p_{1j})} \\ &= 1 + \frac{(p_{11} - 1)\left[\min_{j,i=\overline{2,m}}(p_{ji}) - \max_{j=\overline{2,m}}(p_{1j}) + \prod_{j=2}^{m} p_{1j}\right] - \prod_{j=2}^{m} p_{1j} \max_{j=\overline{2,m}}(p_{j1})}{\min_{j,i=\overline{2,m}}(p_{ji}) - \max_{j=\overline{2,m}}(p_{1j})} > 1. \end{aligned}$$

But, taking φ_1 larger, this implies blow-up of u_1 before $t = 1$; a contradiction. ■

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