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Kernel Polynomial of d-Orthogonal Sequence

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Abstract

The purpose of this work is to construct the kernel polynomial of d-orthogonal sequence. Some properties of this polynomial are established. We prove that this polynomial conserves the $d-$ortohognality, the d-quasi-orthogonality, the d-strictly-orthogonality and the d-weekly-orthogonality.

We finish this work by proving that this polynomial preserved the d-classical and semi-classical orthogonality properties under some conditions.

AMS subject classification:
Keywords: d-orthogonal polynomial- d-quasi-orthogonal polynomial- d-strictly-orthogonal polynomial- classical d-orthgoonal polynomial- semi classical d-orthogonal polynomial- kernel polynomial- Christoffel-Darboux identities.

1. Introduction

In this paper, we construct the kernel polynomial of $d-$orthogonal sequence wich generalizes the result of Chihara [3, 4], Maroni [17], Know and all [12] and recently the main result [21] , in the context of the d-orthogonality.

This construction allows us to write the kernel polynomial of the d-OPS sequence $\{B_n\}_{n\geq 0}$ that we denote by $\left\{B_n^*\right\}_{n\geq 0}$ in the following form

$$B_n^*(x) = \frac{1}{\prod_{i=1}^{d}(x-x_i)}\left[\begin{array}{c} B_{n+d}(x) \\ +\sum_{i=1}^{d}(-1)^i\,\Lambda_{n+d-i}\left(x_1,\ldots,x_d\right)B_{n+d-i}(x)\end{array}\right]$$

where $\Lambda_{n+d-i}(x_1, \ldots, x_d)$ are complex numbers.

We have demonstrated that this Kernel polynomial keeps the d-ortohognality, the d-quasi - orthogonality, the d-strictly - orthogonality and the d-weekly - orthogonality. We finish this work by proving that this polynomial preserves the classical and the semi-classical properties under some conditions that we will be given later.

The paper is divided into four sections. Following the introduction, we give in the second section preliminaries, necessary for the sequel of our work. Construction and definition of Kernel polynomial are established in the third section. Finally, we conclude the paper by giving some properties of this polynomial.

2. Preliminaries and notations

Before discussing our problem, we first give some preliminaries notions which we need below. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$, equipped with its natural inductive limit topology; and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the effect of $u \in \mathcal{P}$ on $f \in \mathcal{P}'$.

In particular, we denote by $(u)_n = \langle u, x^n\rangle$, $n \geq 0$, the moments of u,where $\langle ., .\rangle$ is the dual brakets between the vector space of polynomials with complex coefficients and its dual.

By a polynomial set (PS), we mean a sequence of monic polynomials $\{B_n\}_{n\geq 0}$ which $\deg B_n(x) = n$ for all n, where, $B_n(x) = x^n + \ldots, n \geq 0$.

Let $\{B_n\}_{n\geq 0}$be a polynomial set; there exists a sequence of linear functionals $\{\mathcal{L}_n\}_{n\geq 0}$, such that:

$$\mathcal{L}_n(B_m) = \langle \mathcal{L}_n, B_m\rangle = \delta_{nm} \quad , \quad n, m \geq 0 \tag{2.1}$$

The sequence $\{\mathcal{L}_n\}_{n\geq 0}$ is called the dual sequence of $\{B_n\}_{n\geq 0}$ it is unique [13] .

Lemma 2.1. [13] Let $f \in \mathcal{P}'$ and q be a positive integer. f satisfies

$$f(P_{q-1}) \neq 0 \quad \text{and} \quad f(P_n) = 0, \quad n \geq q$$

if there exist $\lambda_\nu \in \mathbb{C}$, for $0 \leq \nu \leq q-1$, with $\lambda_{q-1} \neq 0$, such that

$$f = \sum_{\nu=0}^{q-1} \lambda_\nu \mathcal{L}_\nu$$

Proposition 2.2. [13] If $\{\mathcal{L}_n\}_{n\geq 0}$ $\left(\text{resp. } \left\{\overset{\frown}{\mathcal{L}}_n\right\}_{n\geq 0}\right)$ is the dual sequence of $\{B_n\}_{n\geq 0}$ $\left(\text{resp. } \{Q_n\}_{n\geq 0}\right)$ (where $(n+1)\, Q_n(x) = DB_{n+1}(x)$) then we have

$$D\overset{\frown}{\mathcal{L}}_n = -(n+1)\, \mathcal{L}_{n+1}, \quad n \geq 0 \tag{2.2}$$

Let us recall the definition of d-orthogonality. We say that the monic sequence $\{B_n\}_{n\geq 0}$ is d-orthogonal polynomial sequence (d-OPS) with respect to the vector of linear functional $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_d)^T$ if it satisfies the following orthogonality relation [7, 8, 20] :

$$\begin{cases} \langle \Gamma_\alpha, x^m B_n(x) \rangle = 0, & n \geq md + \alpha, \quad m \geq 0 \\ \langle \Gamma_\alpha x^m B_{md+\alpha-1}(x) \rangle \neq 0, & m \geq 0 \end{cases} \tag{2.3}$$

or each integer α with $1 \leq \alpha \leq d$. The functionals $\Gamma_1, \Gamma_2, \ldots, \Gamma_d$ are the d first elements of dual sequence $\{\Gamma_n\}_{n\geq 0}$ associated to the sequence of polynomials $\{B_n\}_{n\geq 0}$.

The remarkable characterization of the d-orthogonal sequences is that they check a recurrence of order $d+1$, who is written in the form [20]

$$B_{m+d+1}(x) = (x - \beta_{m+d})B_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} B_{m+d-1-\nu}(x), \quad m \geq 0 \tag{2.4}$$

with the initial conditions

$$B_0(x) = 1, \quad B_1(x) = x - \beta_0$$

and if $d \geq 2$

$$B_n(x) = (x - \beta_{n-1})B_{n-1}(x) - \sum_{\nu=0}^{n-2} \gamma_{n-1-\nu}^{d-1-\nu} B_{n-2-\nu}(x), \quad 2 \leq n \leq d \tag{2.5}$$

and the conditions of regularity:

$$\gamma_{m+1}^0 \neq 0, \quad m \geq 0 \tag{2.6}$$

Definition 2.3. **[20]** A sequence $\{B_n\}_{n\geq 0}$ is said to be d-quasi-orthogonal of order s with respect to $\Gamma = \left(\Gamma^1, \Gamma^2, \ldots, \Gamma^d\right)^T$ if for $1 \leq \alpha \leq d$ there exist $s_\alpha \geq 0$ and $\sigma_\alpha \geq s_\alpha$ integers such that

$$\begin{cases} \langle \Gamma_\alpha, x^m B_n(x) \rangle = 0 \;, \quad n \geq (m + s_\alpha)\, d + \alpha, \quad m \geq 0 \\ \langle \Gamma_\alpha, x^{\sigma_\alpha - s_\alpha} B_{\sigma_\alpha d+\alpha-1}(x) \rangle \neq 0 \end{cases} \tag{2.7}$$

Definition 2.4. **[20]** A sequence $\{B_n\}_{n\geq 0}$ is said to be strictly d-quasi-orthogonal of order s with respect to $\Gamma = \left(\Gamma^1, \Gamma^2, \ldots, \Gamma^d\right)^T$ if it satisfies:

$$\begin{cases} \langle \Gamma^\alpha, x^m B_n(x) \rangle = 0, & n \geq (m + s_\alpha)d + \alpha, \quad m \geq 0 \\ \langle \Gamma^\alpha, x^m B_{(m+s_\alpha)d+\alpha-1}(x) \rangle \neq 0, & m \geq 0 \end{cases} \tag{2.8}$$

for every $1 \leq \alpha \leq d$ with $s = \max_{1\leq\alpha\leq d} s_\alpha$.

Definition 2.5. **[21]** A sequence $\{B_n\}_{n\geq 0}$ is said to be weakly d-orthogonal of index (p, q) with respect to $\Gamma = \left(\Gamma^1, \Gamma^2, \ldots, \Gamma^d\right)^T$ if satisfies for every $1 \leq \beta \leq d$

$$\begin{cases} \langle \Gamma^\beta, B_n(x) \rangle = 0, n \geq p_\beta d + \beta \\ \langle \Gamma^\beta, B_{p_\beta d+\beta-1}(x) \rangle \neq 0 \end{cases} \tag{2.9}$$

where $p = \max\limits_{1\leq\beta\leq d} p_\beta$, and

$$\begin{cases} \left\langle \Gamma^{\beta}, xB_n(x)\right\rangle = 0, \ n \geq \left(q_\beta + 1\right) d + \beta \\ \left\langle \Gamma^{\beta}, B_{(q_\beta+1)d+\beta-1}(x)\right\rangle \neq 0 \end{cases} \tag{2.10}$$

where $q = \max\limits_{1\leq\beta\leq d} q_\beta$.

Remark 2.6. A strictly d-quasi-orthogonal sequences of order p with respect to Γ is d-weakly orthogonal of index $(p, p+1)$ with respect to Γ.

Remark 2.7. If $d = 1$, we have the definition of the weakly orthogonal sequence of index (p, q).

Now, let us introduce the sequence of monic polynomials $\{Q_n\}_{n\geq 0}$ defined by $Q_n = \frac{1}{n+1} D_x B_{n+1}, n \geq 0$ where D_x denotes the derivative operator d/dx. If the sequence $\{Q_n\}_{n\geq 0}$is also d-OPS, the sequence d-orthogonal $\{B_n\}_{n\geq 0}$ is called classical (in the sense of having the Hahn property [8, 9]) and if the sequence $\{Q_n\}_{n\geq 0}$is d-quasi OPS, then the sequence d-orthogonal $\{B_n\}_{n\geq 0}$ is called semi-classical [19].

Remark 2.8. [9] In the case of d-orthogonality (2.2) can be written as

$$D\overset{\backsim *}{\Gamma} = -\Psi(x)\,\Gamma^* \tag{2.11}$$

where

$$\Psi(x) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & d-1 \\ \varphi(x) & \xi_1 & \xi_2 & \cdots & \xi_{d-1} \end{pmatrix} \tag{2.12}$$

with $\deg \varphi(x) = 1$.

Proposition 2.9. [18] Let $\{B_n\}_{n\geq 0}$ be a d-OPS, then it satisfie the generalised Christoffel-Darboux identities

$$\begin{aligned} \left(\prod_{\mu=0}^{n} \gamma_\mu^0\right)^{-1} \begin{vmatrix} B_{n+d}(x_1) & \cdots & B_n(x_1) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ B_{n+d}(x_{d+1}) & & B_n(x_{d+1}) \end{vmatrix} &= \sum_{\nu=0}^{n} (-1)^{(n-\nu)(d-1)+d} \\ \times \left(\prod_{\mu=0}^{\nu} \gamma_\mu^0\right)^{-1} \times \begin{vmatrix} B_{\nu-1+d}(x_1) & \cdots & x_1 B_{\nu-1+d}(x_1) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ B_{\nu-1+d}(x_{d+1}) & \cdots & x_{d+1} B_{\nu-1+d}(x_{d+1}) \end{vmatrix} \end{aligned} \tag{2.13}$$

with $x_i \neq x_j$ if $i \neq j$ and when $\gamma_n^0 \neq 0, \forall n \geq 0\left(\gamma_0^0 = 1\right)$.

3. Construction and definition of Kernel Polynomial

Let $\{B_n\}_{n\geq 0}$ be a d-OPS with respect to the form $\Gamma = \left(\Gamma^1, \Gamma^2, \ldots, \Gamma^d\right)^T$, then the Christoffel-Darboux identity (2.13) for $d \geq 2$ can be written as

$$\frac{\left(\prod_{\mu=0}^{n} \gamma_n^0\right)}{\Theta_n\left(x_2, \ldots, x_d\right)} \sum_{\nu=0}^{n} \frac{(-1)^{k-n}}{\prod_{\mu=0}^{k} \gamma_\mu^0} \sum_{k=0}^{d-1} A_i(x_2, \ldots, x_{d+1}) B_{\nu+k}(x_1) =$$

$$\frac{1}{\prod_{i=2}^{d+1}(x_1 - x_i)} \left[B_{n+d}(x_1) + \sum_{i=1}^{d} (-1)^i \Lambda_{n+d-i}\left(x_2, \ldots, x_{d+1}\right) B_{n+d-i}(x_1) \right]$$

where

$$\Lambda_{n+d-i}\left(x_2, \ldots, x_{d+1}\right) = \frac{\Phi_n\left(x_2, \ldots, x_{d+1}\right)}{\Theta_n\left(x_2, \ldots, x_{d_{+1}}\right)}$$

and

$$\Phi_n\left(x_2, \ldots, x_{d+1}\right) = \begin{vmatrix} B_{n+d}(x_2) & \cdots & B_{n+d-i+1}(x_2) & B_{n+d-i-1}(x_2) & \cdots & B_n(x_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{n+d}(x_{d+1}) & \cdots & B_{n+d-i+1}(x_{d+1}) & B_{n+d-i-1}(x_{d+1}) & \cdots & B_n(x_{d+1}) \end{vmatrix}$$

and

$$\Theta_n\left(x_2, \ldots, x_{d+1}\right) = \begin{vmatrix} B_{n+d}(x_2) & B_{n+d-1}(x_2) & \cdots & B_n(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ B_{n+d}(x_{d+1}) & B_{n+d-1}(x_{d+1}) & \cdots & B_n(x_{d+1}) \end{vmatrix}$$

Definition 3.1. We define a sequence $\left\{B_n^*\right\}_{n\geq 0}$ by

$$B_n^*(x) = \frac{1}{\prod_{i=1}^{d}(x - x_i)} \left[\begin{array}{c} B_{n+d}(x) \\ + \sum_{i=1}^{d} (-1)^i \Lambda_{n+d-i}\left(x_1, \ldots, x_d\right) B_{n+d-i}(x) \end{array} \right] \tag{3.1}$$

and will be called Kernel polynomial of the d-orthogonal sequence $\{B_n\}_{n\geq 0}$.

For any real numbers $x_1, \ldots, x_d$, we consider the new functional Γ^* whose moments of order n are defined by:

$$\Gamma^*\left(x^n\right) = \Gamma_n^* = \Gamma_{n+d} + \sum_{i=1}^{d} \Gamma_{n+d-i}$$

where $\Gamma_n = \Gamma\left(x^n\right)$ is the moment of order n of Γ.

It is evident that for any polynomial $\Pi(x)$ we have

$$\begin{aligned}\Gamma^*[\Pi(x)] &= \Gamma^*\left[\sum_{k=0}^{n} c_k x^k\right] = \sum_{k=0}^{n} c_k \Gamma^*\left[x^k\right] = \sum_{k=0}^{n} c_k \left(\Gamma_{k+d} + \sum_{i=1}^{d} \Gamma_{k+d-i}\right) \\ &= \sum_{k=0}^{n} c_k \Gamma\left[x^{k+d} + \sum_{i=1}^{d} x^{k+d-i}\right] = \sum_{k=0}^{n} c_k \Gamma\left[\prod_{i=1}^{d}(x-x_i)x^k\right] \\ &= \Gamma\left[\prod_{i=1}^{d}(x-x_i)\Pi(x)\right] = \prod_{i=1}^{d}(x-x_i)\Gamma[\Pi(x)]\end{aligned}$$

It is obvious that for any polynomial $\Pi(x)$ of degree n we have

$$\Gamma^*[\Pi(x)] = \prod_{i=1}^{d}(x-x_i)\,\Gamma[\Pi(x)]$$

Lemma 3.2. Let $\{B_n\}_{n\geq 0}$ be a d-OPS with respect to the form $\Gamma = \left(\Gamma^1, \Gamma^2, \ldots, \Gamma^d\right)^T$, then we have

$$\begin{aligned}B_{n+d}(x) &= (n+d+1)\,Q_{n+d}(x) - (n+d)\,(x-\beta_{n+d})\,Q_{n+d-1}(x) \\ &+ \sum_{\nu=0}^{d-1}(n+d-\nu-1)\,\gamma_{n+d-\nu}^{d-1-\nu}\,Q_{n+d-\nu-2}(x)\end{aligned} \tag{3.2}$$

Proof. This is obtained by deriving (2.4). ■

Proposition 3.3. Let $\{B_n\}_{n\geq 0}$ be a d-OPS with respect to the form $\Gamma = \left(\Gamma^1, \Gamma^2, \ldots, \Gamma^d\right)^T$. Then for all real numbers $x_1, \ldots, x_d$ the functional $\Gamma^* = \prod_{i=1}^{d}(x-x_i)\Gamma$ is quasi-defined if:

$$\Lambda_{n+d-i}(x_1, \ldots, x_d) \neq 0 \tag{3.3}$$

In this case, the d-OPS with respect to the form $\Gamma^* = \left(\Gamma_1^*, \Gamma_2^*, \ldots, \Gamma_d^*\right)^T$ is the sequence

$$B_n^*(x) = \frac{1}{\prod_{i=1}^{d}(x-x_i)}\left[\begin{array}{c}B_{n+d}(x) \\ +\sum_{i=1}^{d}(-1)^i\,\Lambda_{n+d-i}(x_1, \ldots, x_d)\,B_{n+d-i}(x)\end{array}\right] \tag{3.4}$$

with

$$\Lambda_{n+d-i}(x_1, \ldots, x_d) = \frac{\Phi_n(x_1, \ldots, x_d)}{\Theta_n(x_1, \ldots, x_d)} \tag{3.5}$$

Proof. We have

$$\left\langle \Gamma_\alpha^*, x^m B_n^*(x) \right\rangle = \left\langle \prod_{i=1}^{d} (x - x_i)\, \Gamma_\alpha, x^m B_n^*(x) \right\rangle$$
$$= \left\langle \Gamma_\alpha, x^m B_{n+d}(x) \right\rangle + \sum_{i=1}^{d} (-1)^{ii}\, \Lambda_{n+d-i}\,(x_1, \ldots, x_d) \left\langle \Gamma_\alpha, x^m B_{n+d-i}(x) \right\rangle$$
$$= 0$$

for $n \geq md + \alpha$ (according to the definition of d-orthogonality of the sequence $\{B_n\}_{n\geq 0}$). And for the regularity we have

$$\left\langle \Gamma_\alpha^*, x^m B_{md+\alpha-1}^*(x) \right\rangle = \left\langle \prod_{i=1}^{d} (x - x_i)\, \Gamma_\alpha, x^m B_{md+\alpha-1}^*(x) \right\rangle$$

$$= \left\langle \Gamma_\alpha, x^m B_{(m+1)d+\alpha-1}(x) \right\rangle$$
$$+ \sum_{i=1}^{d} (-1)^{i}\, \Lambda_{n+d-i}\,(x_1, \ldots, x_d) \left\langle \Gamma_\alpha, x^m B_{(m+1)d+\alpha-i-1}(x) \right\rangle$$

$$= (-1)^d\, \Lambda_{md+\alpha-1}\,(x_1, \ldots, x_d) \left\langle \Gamma_\alpha, x^m B_{md+\alpha-1}(x) \right\rangle \neq 0$$

under the condition

$$\Lambda_{m+d-i}\,(x_1, \ldots, x_d) \neq 0\,, \quad m \geq 0$$

because $\left\langle \Gamma_\alpha, x^m B_{md+\alpha-1}(x) \right\rangle \neq 0$.

From where

$$\begin{cases} \Gamma_\alpha^*(x^m B_n^*(x)) = 0 \;\;,\;\; n \geq md + \alpha \;\;,\;\; m \geq 0 \\ \\ \Gamma_\alpha^*(x^m B_{md+\alpha-1}^*(x)) \neq 0 \;\;,\;\; m \geq 0 \end{cases}$$

i.e $\left\{B_n^*\right\}_{n\geq 0}$ is a d-OPS with respect to the form $\Gamma^* = \prod_{i=1}^{d} (x - x_i)\, \Gamma$ under the condition

$$\Lambda_{m+d-i}\,(x_1, \ldots, x_d) \neq 0\,, \quad m \geq 0$$

■

4. Properties of Kernel Polynomial

The Kernel polynomial $\left\{B_n^*\right\}_{n\geq 0}$ verifies the following properties.

Proposition 4.1. If $\{B_n\}_{n\geq 0}$ is strictly d-quasi-orthogonale of order s with respect to the form Γ, then its Kernel polynomial $\left\{B_n^*\right\}_{n\geq 0}$ is also strictly d-quasi-orthogonale of order s with respect to the form $\Gamma^* = \prod_{i=1}^{d}(x-x_i)\,\Gamma$, under the condition

$$\Lambda_{((s_\alpha+m)d+\alpha-1}(x_1,\ldots,x_d) \neq 0, \quad m \geq 0 \tag{4.1}$$

where $s = \max_{1\leq\alpha\leq d} s_\alpha$.

Proof. We have

$$\left\langle \Gamma_\alpha^*, x^m B_n^*(x)\right\rangle = \left\langle \prod_{i=1}^{d}(x-x_i)\,\Gamma_\alpha, x^m B_n^*(x)\right\rangle$$

$$= \left\langle \Gamma_\alpha, x^m B_{n+d}(x)\right\rangle + \sum_{i=1}^{d}(-1)^i\,\Lambda_{n+d-i}(x_1,\ldots,x_d)\left\langle \Gamma_\alpha, x^m B_{n+d-i}(x)\right\rangle = 0$$

for $n \geq (m+s_\alpha)d+\alpha$ and $m \geq 0$ (according to the definition of the stricly d-orthogonality of the sequence $\{B_n\}_{n\geq 0}$). And for the regularity we have

$$\left\langle \Gamma_\alpha^*, x^m B_{(s_\alpha+m)d+\alpha-1}^*(x)\right\rangle = \left\langle \prod_{i=1}^{d}(x-x_i)\,\Gamma_\alpha, x^m B_{(s_\alpha+m)d+\alpha-1}^*(x)\right\rangle$$
$$= \left\langle \Gamma_\alpha, x^m B_{(s_\alpha+m+1)d+\alpha-1}(x)\right\rangle$$

$$+\sum_{i=1}^{d}(-1)^i\,\Lambda_{(s_\alpha+m+1)d+\alpha-i-1}(x_1,\ldots,x_d)\left\langle \Gamma_\alpha, x^m B_{(s_\alpha+m+1)d+\alpha-i-1}(x)\right\rangle$$

$$= (-1)^d\,\Lambda_{((s_\alpha+m)d+\alpha-1}(x_1,\ldots,x_d)\left\langle \Gamma_\alpha, x^m B_{(s_\alpha+m)d+\alpha-1}(x)\right\rangle \neq 0$$

under the condition

$$\Lambda_{((s_\alpha+m)d+\alpha-1}(x_1,\ldots,x_d) \neq 0, \; m \geq 0$$

Consequently

$$\begin{cases} \left\langle \Gamma_\alpha^*, x^m B_n^*(x)\right\rangle = 0, & n \geq (m+s_\alpha)d+\alpha \quad m \geq 0 \\ \left\langle \Gamma_\alpha^*, x^m B_{(s_\alpha+m)d+\alpha-1}^{*,}(x)\right\rangle \neq 0 & m \geq 0 \end{cases}$$

which define the quasi d-orthogonality of $\left\{B_n^*\right\}_{n\geq 0}$ under the condition

$$\Lambda_{((s_\alpha+m)d+\alpha-1}(x_1,\ldots,x_d) \neq 0, \quad m \geq 0$$

■

Proposition 4.2. If $\{B_n\}_{n\geq 0}$ is weekly d-orthogonal of index (p,q) with respect to the form Γ, then its kernel polynomial $\left\{B_n^*\right\}_{n\geq 0}$ is also weekly d-orthogonal of index (p,q) with respect to the form $\Gamma^* = \prod_{i=1}^{d}(x-x_i)\,\Gamma$, under the conditions

$$\Lambda_{p_\alpha d+\alpha-1}(x_1,\ldots,x_d)\neq 0 \tag{4.2}$$

$$\Lambda_{(q_\alpha+1)d+\alpha-1}(x_1,\ldots,x_d)\neq 0 \tag{4.3}$$

where $p = \max\limits_{1\leq\alpha\leq d} p_\alpha$ and $q = \max\limits_{1\leq\alpha\leq d} q_\alpha$.

Proof. We have

$$\left\langle \Gamma_\alpha^*, B_n^*(x)\right\rangle = \left\langle \prod_{i=1}^{d}(x-x_i)\,\Gamma_\alpha, B_n^*(x)\right\rangle$$

$$= \langle \Gamma_\alpha, B_{n+d}(x)\rangle + \sum_{i=1}^{d}(-1)^i\,\Lambda_{n+d-i}(x_1,\ldots,x_d)\,\langle \Gamma_\alpha, B_{n+d-i}(x)\rangle = 0$$

for $n \geq p_\alpha d+\alpha$ (according to the definition of the weekly d-orthogonality of the sequence $\{B_n\}_{n\geq 0}$) with $p=\max\limits_{\alpha} p_\alpha$. And further

$$\left\langle \Gamma_\alpha^*, B_{p_\alpha d+\alpha-1}^*(x)\right\rangle = \left\langle \prod_{i=1}^{d}(x-x_i)\,\Gamma_\alpha, B_{p_\alpha d+\alpha-1}^*(x)\right\rangle$$

$$= \left\langle \Gamma_\alpha, B_{(p_\alpha+1)d+\alpha-1+d}(x)\right\rangle$$

$$+ \sum_{i=1}^{d}(-1)^i\,\Lambda_{(p_\alpha+1)d+\alpha-1-i}(x_1,\ldots,x_d)\left\langle \Gamma_\alpha, B_{(p_\alpha+1)d+\alpha-1-i}(x)\right\rangle$$

$$= (-1)^i\,\Lambda_{p_\alpha d+\alpha-1}(x_1,\ldots,x_d)\left\langle \Gamma_\alpha, B_{p_\alpha d+\alpha-1}(x)\right\rangle \neq 0$$

under the condition

$$\Lambda_{p_\alpha d+\alpha-1}(x_1,\ldots,x_d)\neq 0$$

In the same way we have

$$\langle \Gamma_\alpha, xB_{n+d}(x)\rangle = 0, \quad n\geq q_\alpha d+\alpha$$

and

$$\left\langle \Gamma_\alpha^*, B_{(q_\alpha+1)d+\alpha-1}^*(x)\right\rangle \neq 0$$

under ther condition

$$\Lambda_{(q_\alpha+1)d+\alpha-1}\left(x_1,\ldots,x_d\right)\neq 0$$

Finally $\left\{B_n^*\right\}_{n\geq 0}$ is d-weekly-orthogonal of index (p,q) with respect to linear form $\Gamma^*=(x-y)(x-z)\,\Gamma$, under the conditions

$$\begin{cases}\Lambda_{p_\alpha d+\alpha-1}\left(x_1,\ldots,x_d\right)\neq 0\\ \Lambda_{(q_\alpha+1)d+\alpha-1}\left(x_1,\ldots,x_d\right)\neq 0\end{cases}$$

■

Proposition 4.3. If $\{B_n\}_{n\geq 0}$ is d-quasi-orthogonal of ordre s with respect to linear form Γ, then its kernel polynomial $\left\{B_n^*\right\}_{n\geq 0}$ is also d-quasi-orthogonal of order s with respect to linear form $\Gamma^*=\prod_{i=1}^{d}(x-x_i)\,\Gamma$, under the conditions

$$\Lambda_{\sigma_\alpha d+\alpha-1}\left(x_1,\ldots,x_d\right)\neq 0 \tag{4.4}$$

Proof. Using the definition of the d-quasi-orthogonality of $\{B_n\}_{n\geq 0}$, we have

$$\left\langle \Gamma_\alpha^*, x^m B_n^*(x)\right\rangle=\left\langle \prod_{i=1}^{d}(x-x_i)\,\Gamma_\alpha, x^m B_n^*(x)\right\rangle$$

$$=\left\langle \Gamma_\alpha, x^m B_{n+d}(x)\right\rangle+\sum_{i=1}^{d}(-1)^i\,\Lambda_{n+d-i}\left(x_1,\ldots,x_d\right)\left\langle \Gamma_\alpha, x^m B_{n+d-i}(x)\right\rangle=0$$

for $n\geq(m+s_\alpha)\,d+\alpha$, $m\geq 0$. And for the regularity we have also

$$\left\langle \Gamma_\alpha^*, x^{\sigma_\alpha-s_\alpha} B_{\sigma_\alpha d+\alpha-1}^*(x)\right\rangle=\left\langle \prod_{i=1}^{d}(x-x_i)\,\Gamma_\alpha, x^{\sigma_\alpha-s_\alpha} B_{\sigma_\alpha d+\alpha-1}^*(x)\right\rangle$$

$$=\left\langle \Gamma_\alpha, x^{\sigma_\alpha-s_\alpha} B_{(\sigma_\alpha+1)d+\alpha-1}(x)\right\rangle$$

$$+\sum_{i=1}^{d}(-1)^i\,\Lambda_{(\sigma_\alpha+1)d+\alpha-1-i}\left(x_1,\ldots,x_d\right)\left\langle \Gamma_\alpha, x^{\sigma_\alpha-s_\alpha} B_{(\sigma_\alpha+1)d+\alpha-1-i}(x)\right\rangle$$

$$=(-1)^d\,\Lambda_{\sigma_\alpha d+\alpha-1}\left(x_1,\ldots,x_d\right)\left\langle \Gamma_\alpha, x^{\sigma_\alpha-s_\alpha} B_{\sigma_\alpha d+\alpha-1}(x)\right\rangle\neq 0$$

if

$$\Lambda_{\sigma_\alpha d+\alpha-1}\left(x_1,\ldots,x_d\right)\neq 0$$

Finally we have

$$\begin{cases}\left\langle \Gamma_\alpha^*, x^m B_n^*(x)\right\rangle=0 & n\geq(m+s_\alpha)\,d+\alpha,\quad m\geq 0\\ \left\langle \Gamma_\alpha^*, x^{\sigma_\alpha-s_\alpha} B_{\sigma_\alpha d+\alpha-1}^*(x)\right\rangle\neq 0, & \end{cases}$$

under the condition $\Lambda_{\sigma_\alpha d+\alpha-1}(x_1,\ldots,x_d)\neq 0$, which gives us the d-quasi-orthogonality of $\left\{B_n^*\right\}_{n\geq 0}$. ■

Notation 4.4. Let be $\overset{\frown}{\Gamma}$ the linear form of the sequence of derivatives

$$\left\{Q_n=\frac{1}{n+1}DB_n\right\}_{n\geq 0}$$

and $\overset{\frown}{\Gamma}^* = \prod_{i=1}^{d}(x-x_i)\,\Gamma^*$ the linear form of the sequence of derivatives

$$\left\{Q_n^*=\frac{1}{n+1}DB_n^*\right\}_{n\geq 0}$$

Proposition 4.5. If $\{B_n\}_{n\geq 0}$ is d-classical OPS with respect to the linear form Γ, then $\left\{B_n^*\right\}_{n\geq 0}$ is also d-classical OPS with respect to the linear form $\Gamma^*=\prod_{i=1}^{d}(x-x_i)\,\Gamma$ under the condition

$$\Lambda_{md+\alpha}(x_1,\ldots,x_d)\neq 0 \tag{4.5}$$

for $m\geq 0$.

Proof. Let $\{B_n\}_{n\geq 0}$be a d-OPS with respect to the linear form Γ, than $\left\{B_n^*\right\}_{n\geq 0}$ is also d-OPS with respect to the linear form $\Gamma^*=\prod_{i=1}^{d}(x-x_i)\,\Gamma$ (by Proposition...). Furthermore the sequence $\left\{Q_n=\frac{1}{n+1}DB_n\right\}_{n\geq 0}$ is also d-OPS with respect to the linear form $\overset{\frown}{\Gamma}$ (according to the definition...). It remains then to show that $\left\{Q_n^*=\frac{1}{n+1}DB_n^*\right\}_{n\geq 0}$ is $d-$OPS with respect to the linear form $\overset{\frown}{\Gamma}^*=\prod_{i=1}^{d}(x-x_i)\,\Gamma^*$.

To begin, we have

$$\left\langle\overset{\frown}{\Gamma}{}^*_\alpha, x^m Q_n^*(x)\right\rangle=\frac{1}{n+1}\left\langle\overset{\frown}{\Gamma}{}^*_\alpha, x^m DB_{n+1}^*(x)\right\rangle$$

$$=\frac{1}{n+1}\left\langle\overset{\frown}{\Gamma}{}^*_\alpha, D\left(x^m B_{n+1}^*(x)\right)\right\rangle-\frac{m}{n+1}\left\langle\overset{\frown}{\Gamma}{}^*_\alpha, x^{m-1}B_{n+1}^*(x)\right\rangle \tag{***}$$

$$=-\frac{1}{n+1}\left\langle D\overset{\frown}{\Gamma}{}^*_\alpha, x^m B_{n+1}^*(x)\right\rangle-\frac{m}{n+1}\left\langle\overset{\frown}{\Gamma}{}^*_\alpha, x^{m-1}B_{n+1}^*(x)\right\rangle$$

and using (2.11) et (2.12) we have

$$\left\langle D\overset{\frown}{\Gamma}_{\alpha}^{*}, x^{m}B_{n+1}^{*}(x)\right\rangle = -\left\langle \Psi(x)\Gamma_{\alpha}^{*}, x^{m}B_{n+1}^{*}(x)\right\rangle$$

$$= -\left\langle \Gamma_{2}^{*}, x^{m}B_{n+1}^{*}(x)\right\rangle - 2\left\langle \Gamma_{3}^{*}, x^{m}B_{n+1}^{*}(x)\right\rangle - ... - \xi_{d-1}\left\langle \Gamma_{d}^{*}, x^{m}B_{n+1}^{*}(x)\right\rangle$$

because

$$\Psi(x)\Gamma_{\alpha}^{*} = \begin{pmatrix} \Gamma_{2}^{*} \\ 2\Gamma_{3}^{*} \\ ... \\ (d-1)\Gamma_{d}^{*} \\ \varphi(x)\Gamma_{1}^{*} + \xi_{1}\Gamma_{2}^{*} + \cdots + \xi_{d-1}\Gamma_{d}^{*} \end{pmatrix}$$

from where

$$\left\langle \Psi(x)\Gamma_{\alpha}^{*}, x^{m}B_{n+1}^{*}(x)\right\rangle = 0, \quad n \geq md + \alpha\ , \quad m \geq 0$$

For the second part of equation$(***)$, we have

$$\left\langle \overset{\frown}{\Gamma}_{\alpha}^{*}, x^{m-1}B_{n+1}^{*}(x)\right\rangle = \left\langle \prod_{i=1}^{d}(x - x_{i})\overset{\frown}{\Gamma}_{\alpha}, x^{m-1}B_{n+1}^{*}(x)\right\rangle$$

$$= \left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1}B_{n+d+1}(x)\right\rangle + \sum_{i=1}^{d}(-1)^{i}\Lambda_{n+d+1-i}(x_{1}, \ldots, x_{d})\left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1}B_{n+d+1-i}(x)\right\rangle$$

and using (3.2) we have

$$\left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1}B_{n+d+1}(x)\right\rangle = (n+d+2)\left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1}Q_{n+d+1}(x)\right\rangle$$

$$-(n+d+1)\left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m}Q_{n+d}(x)\right\rangle + (n+d+1)\beta_{n+d+1}\left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1}Q_{n+d}(x)\right\rangle$$

$$+\sum_{\nu=0}^{d-1}(n+d-\nu)\gamma_{n+d-\nu+1}^{d-1-\nu}\left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1}Q_{n+d-\nu-1}(x)\right\rangle = 0$$

for $n \geq (m-1)d + \alpha + 1$ and $m \geq 0$ (from the definition of d-orthogonality of $\{Q_n\}_{n\geq 0}$).

And in the same manner we have

$$\left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1} B_{n+d+1-i}(x) \right\rangle = (n+d+2-i) \left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1} Q_{n+d+1-i}(x) \right\rangle$$

$$- (n+d+1-i) \left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m} Q_{n+d-i}(x) \right\rangle$$
$$+ (n+d+1-i)\, \beta_{n+d+1-i} \left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1} Q_{n+d-i}(x) \right\rangle$$

$$+ \sum_{\nu=0}^{d-1} (n+d-\nu-i)\, \gamma_{n+d-\nu+1-i}^{d-1-\nu} \left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1} Q_{n+d-\nu-1-i}(x) \right\rangle = 0$$

for $n \geq md + \alpha$ and $m \geq 0$. And therefore we obtain

$$\left\langle \overset{\frown}{\Gamma}{}^{*}_{\alpha}, x^{m} Q_{n}^{*}(x) \right\rangle = 0, \quad n \geq md + \alpha, \quad m \geq 0$$

For the regularity, we have

$$\left\langle \overset{\frown}{\Gamma}{}^{*}_{\alpha}, x^{m} Q_{md+\alpha-1}^{*}(x) \right\rangle = \frac{1}{md+\alpha} \left\langle \overset{\frown}{\Gamma}{}^{*}_{\alpha}, x^{m} D B_{md+\alpha}^{*}(x) \right\rangle$$

$$= \frac{1}{md+\alpha} \left\langle \overset{\frown}{\Gamma}{}^{*}_{\alpha}, D\left(x^{m} B_{md+\alpha}^{*}(x)\right) \right\rangle - \frac{m}{md+\alpha} \left\langle \overset{\frown}{\Gamma}{}^{*}_{\alpha}, x^{m-1} B_{md+\alpha}^{*}(x) \right\rangle$$

and

$$\left\langle \overset{\frown}{\Gamma}{}^{*}_{\alpha}, D\left(x^{m} B_{md+\alpha}^{*}(x)\right) \right\rangle = -\left\langle D\overset{\frown}{\Gamma}{}^{*}_{\alpha}, x^{m} B_{md+\alpha}^{*}(x) \right\rangle = \left\langle \Psi(x)\, \Gamma_{\alpha}^{*}, x^{m} B_{md+\alpha}^{*}(x) \right\rangle$$

$$= \left\langle \Gamma_{2}^{*}, x^{m} B_{md+\alpha}^{*}(x) \right\rangle + 2 \left\langle \Gamma_{3}^{*}, x^{m} B_{md+\alpha}^{*}(x) \right\rangle + \ldots + \xi_{d-1} \left\langle \Gamma_{d}^{*}, x^{m} B_{md+\alpha}^{*}(x) \right\rangle = 0$$

Over there

$$\left\langle \overset{\frown}{\Gamma}{}^{*}_{\alpha}, x^{m-1} B_{md+\alpha}^{*}(x) \right\rangle = \left\langle \prod_{i=1}^{d} (x - x_i)\, \overset{\frown}{\Gamma}_{\alpha}, x^{m-1} B_{md+\alpha}^{*}(x) \right\rangle$$

$$= \left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1} \left[\begin{array}{c} B_{md+\alpha+d}(x) \\ + \sum_{i=1}^{d} (-1)^{i}\, \Lambda_{md+\alpha+d-i}(x_1, \ldots, x_d)\, B_{md+\alpha+d-i}(x) \end{array} \right] \right\rangle$$

$$= \left\langle \overset{\frown}{\Gamma}_{\alpha}, x^{m-1} B_{(m+1)d+\alpha}(x) \right\rangle = 0$$

knowing that

$$
\begin{aligned}
&\left\langle \breve{\Gamma}_\alpha, x^{m-1} B_{(m+1)d+\alpha}(x) \right\rangle = \\
&= ((m+1)\,d+\alpha+1)\left\langle \breve{\Gamma}_\alpha, x^{m-1} Q_{(m+1)d+\alpha+1}\,(x) \right\rangle \\
&-((m+1)\,d+\alpha)\left\langle \breve{\Gamma}_\alpha, x^{m} Q_{(m+1)d+\alpha}\,(x) \right\rangle \\
&+\beta_{(m+1)d+\alpha}\,((m+1)\,d+\alpha)\left\langle \breve{\Gamma}_\alpha, x^{m-1} Q_{(m+1)d+\alpha}\,(x) \right\rangle \\
&+\sum_{\nu=0}^{d-1}((m+1)\,d+\alpha-\nu-i)\,\gamma_{(m+1)d+\alpha-\nu}^{d-1-\nu}\left\langle \breve{\Gamma}_\alpha, x^{m-1} Q_{(m+1)d+\alpha-\nu-2}\,(x) \right\rangle \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=1}^{d}(-1)^i\,\Lambda_{(m+1)d+\alpha-i}\,(x_1,\ldots,x_d)\left\langle \breve{\Gamma}_\alpha, x^{m-1} B_{(m+1)d+\alpha-i}(x) \right\rangle \\
= \; &\sum_{i=1}^{d}(-1)^i\,\Lambda_{(m+1)d+\alpha-i}\,(x_1,\ldots,x_d)\,((m+1)\,d \\
&+\alpha-i+1)\left\langle \breve{\Gamma}_\alpha, x^{m-1} Q_{(m+1)d+\alpha-i+1}\,(x) \right\rangle \\
&-\sum_{i=1}^{d}(-1)^i\,\Lambda_{(m+1)d+\alpha-i}\,(x_1,\ldots,x_d)\,((m+1)\,d \\
&+\alpha-i)\left\langle \breve{\Gamma}_\alpha, x^{m} Q_{(m+1)d+\alpha-i-1}\,(x) \right\rangle \\
&-\sum_{i=1}^{d}(-1)^i\,\Lambda_{(m+1)d+\alpha-i}\,(x_1,\ldots,x_d)\,((m+1)\,d \\
&+\alpha-i)\left\langle \breve{\Gamma}_\alpha, x^{m} Q_{(m+1)d+\alpha-i-1}\,(x) \right\rangle \\
&-\sum_{i=1}^{d}(-1)^i\,\Lambda_{(m+1)d+\alpha-i}\,(x_1,\ldots,x_d)\,((m+1)\,d \\
&+\alpha-i)\left\langle \breve{\Gamma}_\alpha, x^{m} Q_{(m+1)d+\alpha-i-1}\,(x) \right\rangle
\end{aligned}
$$

$$+\sum_{i=1}^{d}(-1)^{i}\Lambda_{(m+1)d+\alpha-i}(x_1,\ldots,x_d)\,\beta_{(m+1)d+\alpha-i}((m+1)d+\alpha-i)$$
$$\times\left\langle\overset{\frown}{\Gamma}_{\alpha},x^{m-1}Q_{(m+1)d+\alpha-i-1}(x)\right\rangle$$
$$+\sum_{i=1}^{d}(-1)^{i}\Lambda_{(m+1)d+\alpha-i}(x_1,\ldots,x_d)\times$$
$$\sum_{\nu=0}^{d-1}((m+1)d+\alpha-\nu-i-1)\,\gamma^{d-1-\nu}_{(m+1)d+\alpha-\nu}\left\langle\overset{\frown}{\Gamma}_{\alpha},x^{m-1}Q_{(m+1)d+\alpha-\nu-2}(x)\right\rangle$$
$$=\;(-1)^{d}\Lambda_{md+\alpha}(x_1,\ldots,x_d)\,\gamma^{0}_{(m-1)d+\alpha+1}\left\langle\overset{\frown}{\Gamma}_{\alpha},x^{m-1}Q_{(m-1)d+\alpha-1}(x)\right\rangle\neq 0$$

under the condition

$$\Lambda_{md+\alpha}(x_1,\ldots,x_d)\neq 0$$

because $\gamma^{0}_{(m-1)d+\alpha+1}\neq 0$. From where

$$\begin{cases}\left\langle\overset{\frown}{\Gamma}{}^{*}_{\alpha},x^{m}Q^{*}_{n}(x)\right\rangle=0, & n\geq md+\alpha,\quad m\geq 0\\ \left\langle\overset{\frown}{\Gamma}{}^{*}_{\alpha},x^{m}Q^{*}_{md+\alpha-1}(x)\right\rangle\neq 0 & m\geq 0\end{cases}$$

this proves that $\left\{Q^{*}_{n}\right\}_{n\geq 0}$is also d- OPS with respect to the linear form $\overset{\frown}{\Gamma}{}^{*}=\prod_{i=1}^{d}(x-x_i)\,\overset{\frown}{\Gamma}$.
This completes the proof. ■

Proposition 4.6. If $\{B_n\}_{n\geq 0}$ is semi-classical d-OPS of ordre s with respect to the linear form Γ, then *its Kernel polynomial* $\left\{B^{*}_{n}\right\}_{n\geq 0}$ is also semi-classical d-OPS of ordre s with respect to the linear form $\Gamma^{*}=\prod_{i=1}^{d}(x-x_i)\,\Gamma$ under the condition

$$\sigma_{\alpha}\neq s_{\alpha} \tag{4.6}$$

and

$$\Lambda_{\sigma_{\alpha}d+\alpha}(x_1,\ldots,x_d)\neq 0 \tag{4.7}$$

for each integer α with $1\leq\alpha\leq d$.

Proof. For the proof it suffices to show that the sequence $\left\{Q^{*}_{n}=\frac{1}{n+1}DB^{*}_{n}\right\}_{n\geq 0}$ is d-quasi-orthogonal of order s with respect to the linear form $\widetilde{\Gamma}^{*}=\prod_{i=1}^{d}(x-x_i)\,\widetilde{\Gamma}$.For this we have

$$\left\langle \overset{\sim *}{\Gamma}_{\alpha}, x^{m} Q_{n}^{*}(x) \right\rangle = \frac{1}{n+1} \left\langle \overset{\sim *}{\Gamma}_{\alpha}, x^{m} D B_{n+1}^{*}(x) \right\rangle$$
$$= \frac{1}{n+1} \left\langle \overset{\sim *}{\Gamma}_{\alpha}, D \left(x^{m} B_{n+1}^{*}(x) \right) \right\rangle - \frac{m}{n+1} \left\langle \overset{\sim *}{\Gamma}_{\alpha}, x^{m-1} B_{n+1}^{*}(x) \right\rangle$$
$$= -\frac{1}{n+1} \left\langle D \overset{\sim *}{\Gamma}_{\alpha}, x^{m} B_{n+1}^{*}(x) \right\rangle - \frac{m}{n+1} \left\langle \overset{\sim *}{\Gamma}_{\alpha}, x^{m-1} B_{n+1}^{*}(x) \right\rangle$$

using (2.11) et (2.12) we have

$$\left\langle D \overset{\sim *}{\Gamma}_{\alpha}, x^{m} B_{n+1}^{*}(x) \right\rangle = 0, n \geq (m + s_{\alpha}) d + \alpha, \quad m \geq 0$$

and

$$\left\langle \overset{\sim *}{\Gamma}_{\alpha}, x^{m-1} B_{n+1}^{*}(x) \right\rangle = \left\langle \prod_{i=1}^{d} (x - x_{i}) \tilde{\Gamma}_{\alpha}, x^{m-1} B_{n+1}^{*}(x) \right\rangle$$
$$= \left\langle \tilde{\Gamma}_{\alpha}, x^{m-1} B_{n+d+1}(x) \right\rangle + \sum_{i=1}^{d} (-1)^{i} \Lambda_{n+d+1-i}(x_{1}, \ldots, x_{d}) \left\langle \tilde{\Gamma}_{\alpha}, x^{m-1} B_{n+d+1-i}(x) \right\rangle$$

using (3.2) we obtain

$$\left\langle \tilde{\Gamma}_{\alpha}, x^{m-1} B_{n+d+1}(x) \right\rangle = (n+d+2) \left\langle \tilde{\Gamma}_{\alpha}, x^{m-1} Q_{n+d+1}(x) \right\rangle$$
$$- (n+d+1) \left\langle \tilde{\Gamma}_{\alpha}, x^{m} Q_{n+d}(x) \right\rangle + (n+d+1) \beta_{n+d+1} \left\langle \tilde{\Gamma}_{\alpha}, x^{m-1} Q_{n+d}(x) \right\rangle$$
$$+ \sum_{\nu=0}^{d-1} (n+d-\nu) \gamma_{n+d-\nu+1}^{d-\nu-1} \left\langle \tilde{\Gamma}_{\alpha}, x^{m-1} Q_{n+d-\nu-1}(x) \right\rangle = 0$$

for

$$n \geq (m + s_{\alpha} - 1) d + \alpha, \quad m \geq 0$$

And in the same way we have for $1 \leq i \leq d$

$$\sum_{i=1}^{d} (-1)^i \, \Lambda_{n+d+1-i} \, (x_1, \ldots, x_d) \left\langle \widetilde{\Gamma}_\alpha, x^{m-1} B_{n+d+1-i}(x) \right\rangle =$$
$$\sum_{i=1}^{d} (-1)^i \, \Lambda_{n+d+1-i} \, (x_1, \ldots, x_d) \times$$
$$\left[\begin{array}{l} (n+d+2-i) \left\langle \widetilde{\Gamma}_\alpha, x^{m-1} Q_{n+d+1-i} \, (x) \right\rangle \\ -(n+d+1-i) \, (x - \beta_{n+d+1-i}) \left\langle \widetilde{\Gamma}_\alpha, x^{m-1} Q_{n+d-i} \, (x) \right\rangle \\ + \sum_{\nu=0}^{d-1} (n+d-\nu-i) \, \gamma_{n+d-i-1-\nu}^{d-\nu-1} \left\langle \widetilde{\Gamma}_\alpha, x^{m-1} Q_{n+d-i-\nu-1} \, (x) \right\rangle \end{array} \right] = 0$$

for

$$n \geq (m + s_\alpha) \, d + \alpha, \quad m \geq 0$$

which gives

$$\left\langle \widetilde{\Gamma}^*_\alpha, x^m \, Q_n^*(x) \right\rangle = 0, \quad n \geq (m + s_\alpha) \, d + \alpha, \quad m \geq 0$$

For the regularity, $\exists s_\alpha \geq 0$ et $\sigma_\alpha \geq s_\alpha$ such as

$$\left\langle \widetilde{\Gamma}^*_\alpha, x^{\sigma_\alpha - s_\alpha} \, Q^*_{\sigma_\alpha d + \alpha - 1}(x) \right\rangle = \frac{1}{\sigma_\alpha d + \alpha} \left\langle \widetilde{\Gamma}^*_\alpha, x^{\sigma_\alpha - s_\alpha} \, D B^*_{\sigma_\alpha d + \alpha}(x) \right\rangle$$
$$= \frac{1}{\sigma_\alpha d + \alpha} \left\langle \widetilde{\Gamma}^*_\alpha, D \left[x^{\sigma_\alpha - s_\alpha} B^*_{\sigma_\alpha d + \alpha}(x) \right] \right\rangle - \frac{\sigma_\alpha - s_\alpha}{\sigma_\alpha d + \alpha} \left\langle \widetilde{\Gamma}^*_\alpha, x^{\sigma_\alpha - s_\alpha - 1} B^*_{\sigma_\alpha d + \alpha}(x) \right\rangle$$

and in the same manner as previously we have

$$\begin{aligned} \left\langle \widetilde{\Gamma}^*_\alpha, D \left[x^{\sigma_\alpha - s_\alpha} B^*_{\sigma_\alpha d + \alpha}(x) \right] \right\rangle &= -\left\langle D \widetilde{\Gamma}^*_\alpha, x^{\sigma_\alpha - s_\alpha} B^*_{\sigma_\alpha d + \alpha}(x) \right\rangle \\ &= \left\langle \Psi \, (x) \, \Gamma^*_\alpha, x^{\sigma_\alpha - s_\alpha} B^*_{\sigma_\alpha d + \alpha}(x) \right\rangle \\ &= 0 \end{aligned}$$

et

$$
\begin{aligned}
& \frac{\sigma_\alpha - s_\alpha}{\sigma_\alpha d + \alpha}\left\langle \widetilde{\Gamma}^*_\alpha, x^{\sigma_\alpha - s_\alpha - 1} B^*_{\sigma_\alpha d+\alpha}(x)\right\rangle \\
= & \frac{\sigma_\alpha - s_\alpha}{\sigma_\alpha d + \alpha}\left\langle \prod_{i=1}^{d}(x - x_i)\,\widetilde{\Gamma}_\alpha, x^{\sigma_\alpha - s_\alpha - 1} B^*_{\sigma_\alpha d+\alpha}(x)\right\rangle \\
= & \frac{\sigma_\alpha - s_\alpha}{\sigma_\alpha d + \alpha}\left\langle \widetilde{\Gamma}_\alpha, x^{\sigma_\alpha - s_\alpha - 1} B_{\sigma_\alpha d+\alpha+d}(x)\right\rangle \\
& + \frac{\sigma_\alpha - s_\alpha}{\sigma_\alpha d + \alpha}^{\sigma_\alpha - s_\alpha - 1} \sum_{i=1}^{d} (-1)^i\, \Lambda_{\sigma_\alpha d+\alpha+d-i}\,(x_1, \ldots, x_d)\left\langle \widetilde{\Gamma}_\alpha, x B_{\sigma_\alpha d+\alpha+d-i}(x)\right\rangle \\
= & \frac{\sigma_\alpha - s_\alpha}{\sigma_\alpha d + \alpha}\sum_{i=1}^{d} (-1)^i\, \Lambda_{\sigma_\alpha d+\alpha+d-i}\,(x_1, \ldots, x_d)\left\langle \widetilde{\Gamma}_\alpha, x^{\sigma_\alpha - s_\alpha - 1} Q_{\sigma_\alpha d+\alpha+d-i}(x)\right\rangle \\
= & (-1)^d\,\frac{\sigma_\alpha - s_\alpha}{(\sigma_\alpha d + \alpha)}\,(\sigma_\alpha d + \alpha)\,\Lambda_{\sigma_\alpha d+\alpha}\,(x_1, \ldots, x_d)\left\langle \widetilde{\Gamma}_\alpha, x^{\sigma_\alpha - s_\alpha} Q_{\sigma_\alpha d+\alpha-1}(x)\right\rangle \\
= & (-1)^d\,(\sigma_\alpha - s_\alpha)\,\Lambda_{\sigma_\alpha d+\alpha}\,(x_1, \ldots, x_d)\left\langle \widetilde{\Gamma}_\alpha, x^{\sigma_\alpha - s_\alpha} Q_{\sigma_\alpha d+\alpha-1}(x)\right\rangle \\
\neq & \; 0
\end{aligned}
$$

under the condition

$$(\sigma_\alpha - s_\alpha)\,\Lambda_{\sigma_\alpha d+\alpha}\,(x_1, \ldots, x_d) \neq 0$$

i.e.

$$\sigma_\alpha \neq s_\alpha$$

and

$$\Lambda_{\sigma_\alpha d+\alpha}\,(x_1, \ldots, x_d) \neq 0$$

this completes the proof. ■

Remark 4.7. For $d = 1$ we find the kernel polynomial obtained by Chihara [3, 4, 12]. For $d = 2$ we find the kernel polynomial of $2-$ orthogonal sequence obtained in [21].

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