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On the solutions for neutral fractional differential equations with state dependent-delay

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Abstract

In this manuscript, by utilizing Monch fixed point theorem and the techniques of measure of noncompactness, the existence results for semilinear neutral fractional differential equations with state dependent-delay involving the Riemann-Liouville fractional derivative are investigated. Our approaches also refers resolvent operators theory. To illustrate the application of the obtained results an example is provided.

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1. Introduction

Fractional differential equations have become an important object of an investigation in the recent decades and achieved great progress, due to their broad applications in science and engineering such as physics, mechanics, biology, economics,engineering and medical domains. Compared with ordinary and partial differential systems, fractional differential equation systems have the strong potential to modulate the real world problems with high accuracy. In order to further study of these models, it is very much need to the study of fractional differential equations analytically. With the aim of analyzing fractional differential systems for the above, extensive investigations had been carried out. For more details the theory and applications in this field, see the monographs of Kilbas et al. [16], Miller and Ross [18], Podlubny [20], Lakshmikantham [17], Baleanu et al. [7], Zhou [27] and the reference there in.

On the other hand, fractional differential equation with state dependent-delay appears frequently in applications as models of equations. Investigation of these classes of delay equations essentially differ from once of equations with constant or time dependent delay. For the so called reasons, the theory of fractional differential equations with state dependent delay has drawn the attention of researchers in the recent years, see for [3, 11, 13] instance. Moreover the Cauchy problem for various delay equations in Banach spaces has been receiving more and more attention during the past decays see for instance [4, 5, 10, 22, 23, 25] and references cited there in.

Recently, Benchohra [11] *et al.* studied the existence results for some neutral partial functional differential equations of fractional order with state-dependent delay using semi norms in Frechet spaces combined with α-resolvent family. Toufik *et al.* [13] established the existence of mild solutions for impulsive fractional stochastic differential inclusions with state-dependent delay using the nonlinear alternative of Leray-Schauder type for multivalued maps due to O'Regan. Several works was published on the existence of mild solutions for this type of problems using different fixed point techniques.

Motivated by the above literatures [3, 5, 11, 13, 21], the purpose of this article is to establish the existence of mild solutions of neutral fractional differential equations of fractional order with state dependent-delay in Banach space when the delay is finite.

The paper is organized as follows. In section 2, we will briefly recall some preliminary facts which will be used in the paper. Section 3 is devoted to the existence of mild solution using measure of noncompactness and Monch fixed point theorem. The application of our result is given in section 4. We consider the following initial value problem

$$D^{\alpha}\left[y(t)-g\left(t,\, y\left(t-\rho\left(y\left(t\right)\right)\right)\right)\right]=Ay(t)+f\left(t,\, y\left(t-\rho\left(y\left(t\right)\right)\right)\right),\quad t\in[0,\, b]\,,\ 0<\alpha<1 \tag{1.1}$$

$$y\left(0\right)\ =\varphi\in\mathscr{B},\ t\in[-r,\, 0] \tag{1.2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, f and $g : J \times C([-r, 0], E) \to E$ are continuous functions $A : D(A) \subset E \to E$ is a densely defined closed linear operator on E, $\varphi : [-r\ ,\ 0] \to E$ a given continuous function with $\varphi\,(0)-g\,(0,\ \varphi)=0$ and $(E, |.|)$ a real Banach space, ρ is a positive bounded continuous function on $C([-r, 0], E)$. r is the maximal delay defined by

$$r = \sup_{y \in c} \rho(y).$$

Moreover any element in $\mathscr{B}$ is given by $x_t(\theta) = x(t + \theta)$ axiomatically, where $\theta \in [-r, 0]$ belongs to some abstract phase space $\mathscr{B}$.

2. Some Background Results

In this section, we collect few definitions and propositions of fractional calculus theory and resolvent operators which will be needed in the sequel. Let E be a Banach space and $C([0, b], E)$ represents the Banach space of all continuous functions from [0,b] in to E, normed by

$$\|y\|_{\infty} = \sup_{t \in [0,b]} |y(t)|.$$

$C([0, b], E)$ is endowed with norm defined by

$$\|\phi\|_{C} = sup\{|\phi(\theta)| : \theta \in [-r, 0]\}.$$

$\mathscr{B}(E)$ is the space of bounded linear operators from E in to E, with norm

$$\|N\|_{\mathscr{B}(E)} = sup\{|N(y)| : |y| = 1\}.$$

A measurable function $Y : [0, b] \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. Let $L'([0, b], E)$ denotes the Banach space of measurable functions normed by

$$\|y\|_{L'} = \int_0^b |y(t)|\, dt.$$

$L^{\infty}([0, b], E)$ denotes the Banach space of measurable functions $y : [0, b] \to E$ which are bounded, equipped with the norm

$$\|y\|_{L^{\infty}} = \inf\{k > 0 : \|y\| < k,\ a.e,\ t \in [0, b]\}.$$

For a given set W of functions $w(t) : [-r, b] \to E$, let us denote by

$$W(t) = \{w(t) : w \in W\},\ t \in [-r, b] \quad \text{and}$$
$$W(j) = \{w(t) : w \in W,\ t \in [-r, b]\}.$$

Definition 2.1. [16, 20] The Riemann-Liouville fractional primitive of order $\alpha \in R^+$ of a function $f : [0, b] \to E$ is defined by

$$I_0^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds,$$

provided the right hand side is point-wise defined on $(0, b]$.

Definition 2.2. [16, 20] The Riemann-Liouville derivative of order $0 < \alpha < 1$ of a continuous function $f : [0, b] \to E$ is defined by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{n-q-1} f(s) ds = \frac{d}{dt} I_0{}^{1-\alpha} f(t).$$

Definition 2.3. A map $f : [0, b] \times C([0, b], E) \to E$ is said to Caratheodory if

i) $t \to f(t, u)$ is measurable for each $u \in C([-r, 0], E)$;

ii) $u \to F(t, u)$ is continuous for almost each $t \in J$.

Consider the fractional differential equation

$$D^\alpha y(t) = Ay(t) + f(t), \qquad t \in [0, b], \quad 0 < \alpha < 1, \qquad y(0) = 0, \qquad (2.1)$$

where A is a closed linear unbounded operator in E and $f \in C(J, E)$. Equation (2.1) is equivalent to the following integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} A \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds. \qquad (2.2)$$

The above equation can be written in the following form of integral equation

$$y(t) = h(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A\, y(s)\, ds \quad t \geq 0, \qquad (2.3)$$

where

$$h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds. \qquad (2.4)$$

The exact solution of (2.1) and the integral equation (2.2) are same given in [16]. Let us assume that the integral equation (2.3) has an resolvent operator $S(t), t \geq 0$ on E.

Next we define the resolvent operator for equation (2.3).

Definition 2.4. [14, Definition 1.1.3] A one parameter family of bounded linear operator $S(t), t \geq 0$ on E is called a resolvent operator for (2.2) if the following conditions are hold.

(i) $S(\cdot)x \in C([0,\infty), E)$ and $S(0)x = x$ for all $x \in E$.

(ii) $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and every $t \geq 0$;

(iii) for every $x \in D(A)$ and $t \geq 0$,

$$S(t)x = x + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A\, S(s)x ds. \tag{2.5}$$

We assume that the resolvent operator $S(t), t \geq 0$ is analytic and there exists a function $\zeta_A \in L'_{loc}([0,\infty), R^+)$ such that $\|S'(t)x\| \leq \zeta_A(t)\|x\|_{[D(A)]}$ for all $t \geq 0$, and each $x \in D(A)$.

Definition 2.5. A function $u \in C(J, E)$ is called a mild solution of the integral equation (2.3) on J if $\int_0^t (t-s)^{\alpha-1}u(s)ds \in D(A)$ for all $t \in J$, $h(t) \in C(J, E)$ and

$$u(t) = \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}u(s)ds + h(t), \text{ for all } t \in J.$$

Lemma 2.6. [14] Under the above said conditions, the following properties are valid.

(i) If $u\,(\cdot)$ is a mild solution of (2.3) on J, then the function $t \to \int_0^t S(t-s)h(s)ds$ is continuously differentiable on J and

$$u(t) = \frac{d}{dt} \int_0^t s(t-s)h(s)ds, \text{ for all } t \in J.$$

(ii) If $h \in C^\beta(J, E)$ for some $\beta \in (0, 1)$, then the function defined by

$$u(t) = S(t)(h(t) - h(0)) + \int_0^t s'(t-s)[h(s) - h(t)]ds + S(t)h(0), \ t \in J$$

is a mild solution of (2.3) on J.

(iii) If $h \in C(J, [D(A)])$ then the function $u : J \to E$ defined by

$$u(t) = \int_0^t s'(t-s)h(s)ds + h(t), \ t \in J$$

is a mild solution of (2.3) on E.

Definition 2.7. Let E be a Banach space and Ω_E the bounded subsets of E. The *Kuratowski measures of noncompactness* is the map $\alpha : \Omega_E \to [0, \infty]$ defined by

$$\alpha(\mathrm{B}) = inf\left\{\varepsilon > 0; \mathrm{B} \subseteq \bigcup_{i=1}^{n} \mathrm{B}_i \text{ and } diam\,(\mathrm{B}_i) \leq \varepsilon\right\} \text{ where } \mathrm{B} \in \Omega_E.$$

The Kuratowski measures of noncompactness satisfies the following properties.

(a) $\alpha(\mathrm{B}) = 0 \Leftrightarrow \bar{\mathrm{B}}$ is compact

(b) $\alpha(\mathrm{B}) = \alpha(\bar{\mathrm{B}})$

(c) $\mathrm{A} \subset \mathrm{B} \Rightarrow \alpha(\mathrm{A}) \leq \alpha(\mathrm{B})$

(d) $\alpha(\mathrm{A} + \mathrm{B}) \leq \alpha(\mathrm{A}) + \alpha(\mathrm{B})$

(e) $\alpha(K\mathrm{B}) = |K|\alpha(\mathrm{B});\ K \in R$

(f) $\alpha(conv\mathrm{B}) = \alpha(\mathrm{B})$.

Theorem 2.8. [2, 19] Let V be a bounded, closed and convex subset of a Banach space such that $0 \in V$, and let N be a continuous mapping of V in to itself. If the implication

$$U = \overline{conv}N(u) \text{ or } U = N(u) \cup \{0\} \Rightarrow \alpha(U) = 0$$

holds for every subset U of V, then N has a fixed point.

Lemma 2.9. [22] Let V be a bounded, closed and convex subset of the Banach space C(J,E), F is a continuous function on $J \times J$ and f : $J \times C([-r, 0), E) \to E$ which satisfies the Caratheodory conditions and there exists $p \in L^1(J, R_+)$ such that for each $t \in J$ and each bounded set $B \subset C([-r, 0], E)$ we have

$$\lim_{k \to 0^+} \alpha(f(J_{t,k} \times B)) \leq p(t)\alpha(B), \text{ where } J_{t,k} = [t - k, t] \cap J.$$

If U is an equicontinuous subset of V, then

$$\alpha\left(\left\{\int_J F(s,t) f(s, y_s) ds;\ y \in U\right\}\right) \leq \int_J \|F(t,s)\| p(s) \alpha\,(U\,(s))\,ds.$$

Remark 2.10. The system (1.1)-(1.2) is equivalent to the following integral equation

$$y(t) = g(t, y(t - \rho(y(t)))) + \frac{A}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s-\rho(y(s))))ds, \quad t \in J$$

stimulated by Lemma (2.1) and the above integral representation, we introduce the concept of mild solution.

Definition 2.11. We say that a continuous function $y : [-r, b] \to E$ is a mild solution of the system (1.1)-(1.2) if

(i) $\int_0^t (t-s)^{\alpha-1} y(s)ds \in D(A)$, where $t \in J$.

(ii) $y(t) = \phi(t), \ t \in [-r, 0]$,

(iii)

$$y(t) = g(t, y(t-\rho(y(t)))) + \frac{A}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y(s)ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s-\rho(y(s))))ds, \quad t \in J$$

(iv) $y(0) - g(0, y(0-\rho(y(0)))) = 0$,

suppose that there exist a resolvent operator $S(t), t \geq 0$ which is differentiable and the functions f and g are continuous. Then by Lemma (2.1) (iii), if $y : [-r, b] \to E$ is a mild solution of the system (1.1)-(1.2), then

$$y(t) = g(t, y(t-\rho(y(t)))) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s-\rho(y(s))))ds + \int_0^t S'(t-s)$$
$$\times \left\{ g(s, y(s-\rho(y(s)))) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\tau)^{\alpha-1} f(\tau - y(\tau-\rho(y(\tau))))d\tau \right\} ds,$$
$$t \in [0, b].$$

The following hypothesis are introduced for further applications.

H1 The operator $S'(t)$ is compact for all $t \geq 0$; and for all $t > 0$ and each $x \in D(A)$

$$\left\| S'(t)x \right\| \leq \zeta_A(t)\|x\|_{[D(A)]}$$

H2 Both f and g : $J \times C([-r, 0]) \to E$ are Caratheodory.

H3 There exist functions $p, q \in L^{\infty}(J, +)$ such that

$$\|f(t, u)\| \leq p(t)\,(\|u\|_C + 1)$$

and

$$\|g(t, u)\| \leq q(t)\,(\|u\|_C + 1)$$

for almost everywhere $t \in J$ and $u \in C([-r, 0], E)$.

H4 For all $L > 0$ such that

$$|g(t, u) - g(t, v)| \leq L\|u - v\|_{\mathscr{B}}$$

for all $t \in [0, b]$ and $u, v \in \mathscr{B}$

H5 For almost each $t \in J$ and each bounded set $\mathscr{B} \subset C([-r, 0], E)$ we have

$$\lim_{k \to 0^+} \alpha\left(f(J_{t,k} \times B)\right) \leq p(t)\alpha(B)$$

and

$$\lim_{k \to 0^+} \alpha\left(g(J_{t,k} \times B)\right) \leq q(t)\alpha(B).$$

3. Existence results

In this section, we will prove the existence of mild solutions of the system (1.1)-(1.2).

Theorem 3.1. Assume that (**H1**)-(**H5**) holds, then the system (1.1)-(1.2) has atleast one mild solution on $[-r, b]$ provided that

$$\frac{b^{\alpha}\|p\|_{L^{\infty}}\left(1 + \|\zeta(A)\|_{L^1}\right) + \Gamma(\alpha + 1)\|\zeta(A)\|_{L^1}\|q\|_{L^{\infty}}}{\Gamma(\alpha + 1)} < 1. \tag{3.1}$$

Proof. Convert the system (1.1)-(1.2) in to a fixed point problem. For that consider the operator $\Psi : C([-r, b], E) \to C([-r, b], E)$ by

$$\Psi(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ g(t, y(t - \rho(y(t)))) + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t - s)^{\alpha - 1} f(s, y(s - \rho(y(s))))ds & \\ + \displaystyle\int_0^t S'(t - s)\left\{\begin{array}{l} g(s, y(s - \rho(y(s)))) \\ + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^s (s - \tau)^{\alpha - 1} f(\tau - y(\tau - \rho(y(\tau))))d\tau \end{array}\right\} ds, & t \in J. \end{cases}$$

Let $\delta > 0$ be such that

$$\delta \geq \left(\|q\|_{L^\infty} + \frac{b^\alpha \|p\|_{L^\infty}}{\Gamma(\alpha + 1)} \right)$$

and consider the set

$$U_\delta = \left\{ y \in C([-r, b], E); \; \|y\|_\infty \leq \delta \right\}.$$

The subset U_δ is closed, bounded and convex. We shall show that Ψ satisfies the assumptions of Theorem (3.1). For the purpose of proving Ψ is completely continuous, divide the operator Ψ into two operators:

$$\Psi_1(y)\,(t) = g(t, y(t - \rho(y(t)))) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, y(s - \rho(y(s))))ds$$

and $\quad \Psi_2(y)(t) = \int_0^t S'(t - s)\Psi_1(y)(s)ds$. We have to show that Ψ_1 and Ψ_2 are completely continuous.

Step 1: In order to prove Ψ_1 is completely continuous, at first we show that Ψ_1 is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ as $n \to \infty$ in $C([-r, b], E)$, then for any $t \in [0, b]$ and $-r \leq s - \rho(y(s)) \leq s$ for every $s \in J$, we get

$$\begin{aligned} &|\Psi_1(y_n)\,(t) - \Psi_1(y)\,(t)| \\ &= \left| \begin{array}{l} g(t, y_n(t - \rho(y_n(t)))) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t - s)^{\alpha - 1} f(s, y_n(s - \rho(y_n(s))))ds \\ -g(t, y(t - \rho(y(t)))) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t - s)^{\alpha - 1} f(s, y(s - \rho(y(s))))ds \end{array} \right| \\ &\leq |g(t, y_n(t - \rho(y_n(t)))) - g(t, y(t - \rho(y(t))))| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \left| \int_0^t (t - s)^{\alpha - 1} f(s, y_n(s - \rho(y_n(s)))) - f(s, y(s - \rho(y(s)))) \right| ds, \end{aligned}$$

since f and g are Caratheodory functions for $t \in J$, and from the continuity of ρ, we have by dominated convergence theorem, the right hand side of the above inequality tends to zero as $n \to \infty$. Therefore $|\Psi_1(y_n) - \Psi_1(y)\,|_\infty \to 0$ as $n \to \infty$. This implies that Ψ_1 is continuous. Next, we will prove that Ψ_1 is bounded. For each $y \in U_\delta$ by (**H3**) and (3.1), for each $t \in J$, we have

$$|\Psi_1(y)\,(t)| = \left| g(t, y(t - \rho(y(t)))) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, y(s - \rho(y(s))))ds \right|$$

$$\leq \|q\|_{L^\infty} (\|y(t)\| + 1) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)\ (\|y(s)\| + 1)\, ds$$
$$\leq \|q\|_{L^\infty} (\delta + 1) + \frac{(\delta+1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)\, ds$$
$$\leq \|q\|_{L^\infty} (\delta + 1) + \frac{(\delta+1)}{\Gamma(\alpha)} \|p\|_{L^\infty} \frac{b^\alpha}{\alpha}$$
$$= (\delta+1) \left\{ \|q\|_{L^\infty} + \frac{\|p\|_{L^\infty} b^\alpha}{\Gamma(\alpha+1)} \right\}$$
$$\leq \delta.$$

Then $\Psi_1(U_\delta) \subset U_\delta$. Now, we shall show that $\Psi_1(U_\delta)$ is equicontinuous. Let η_1 and $\eta_2 \in J,\ \eta_2 > \eta_1$. Then if $\varepsilon > 0$ and $\varepsilon \leq \eta_1 \leq \eta_2$, we have for any $y \in U_\delta$;

$$|\Psi_1(y)(\eta_2) - \Psi_1(y)(\eta_1)|$$
$$= \left| \begin{matrix} g(\eta_2,\ y(\eta_2 - \rho(y(\eta_2)))) + \frac{1}{\Gamma(\alpha)} \int_0^{\eta_2} (\eta_2 - s)^{\alpha-1} f(s,\ y(s - \rho(y(s))))\, ds \\ -g(\eta_1,\ y(\eta_1 - \rho(y(\eta_1)))) + \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} (\eta_1 - s)^{\alpha-1} f(s,\ y(s - \rho(y(s))))\, ds \end{matrix} \right|$$
$$\leq L\ \|\eta_2 - \eta_1\| + \left| \frac{1}{\Gamma(\alpha)} \int_0^{\eta_2} (\eta_2 - s)^{\alpha-1} f(s,\ y(s - \rho(y(s))))\, ds \right|$$
$$- \left| \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1} (\eta_1 - s)^{\alpha-1} f(s,\ y(s - \rho(y(s))))\, ds \right|$$
$$\leq L\ \|\eta_2 - \eta_1\| + \left| \frac{1}{\Gamma(\alpha)} \int_0^{\eta_1-\varepsilon} [(\eta_2 - s)^{\alpha-1} - (\eta_1 - s)^{\alpha-1}] f(s,\ y(s - \rho(y(s))))\, ds \right|$$
$$+ \left| \frac{1}{\Gamma(\alpha)} \int_{\eta_1-\varepsilon}^{\eta_1} [(\eta_2 - s)^{\alpha-1} - (\eta_1 - s)^{\alpha-1}] f(s,\ y(s - \rho(y(s))))\, ds \right|$$
$$+ \left| \frac{1}{\Gamma(\alpha)} \int_{\eta_1}^{\eta_2} (\eta_2 - s)^{\alpha-1} f(s,\ y(s - \rho(y(s))))\, ds \right|$$

$$\leq L\ \|\eta_2 - \eta_1\| + \frac{(\delta+1)\,\|p\|_{L^\infty}}{\Gamma(\alpha)} \left[\begin{array}{l} \int\limits_0^{\eta_1-\varepsilon} [(\eta_2 - s)^{\alpha-1} - (\eta_1 - s)^{\alpha-1}]\,ds \\ + \int\limits_{\eta_1-\varepsilon}^{\eta_1} [(\eta_2 - s)^{\alpha-1} - (\eta_1 - s)^{\alpha-1}]\,ds \\ + \int\limits_{\eta_1}^{\eta_2} (\eta_2 - s)^{\alpha-1}\,ds \end{array} \right]$$

as ε, sufficiently small, and $\eta_1 \to \eta_2$, the right hand side of the above inequality tends to zero. Therefore $\Psi_1(U_\delta)$ is completely continuous.

Step 2: To show that Ψ_2 is completely continuous. Since $S'(\cdot) \in ([0, b], E)$ and $\Psi_1(U_\delta)$ is completely continuous. So it is clear that Ψ_2 is continuous as we proved in step 1. For any $y \in U_\delta$, we have

$$\begin{aligned} |\Psi_2(y)(t)| &\leq \int\limits_0^t \left|S'(t-s)\right| |\Psi_1(y)(s)|\,ds \\ &\leq \int\limits_0^t \zeta_A(t-s)\|\Psi_1(y)(s)\|_{[D(A)]}ds \\ &\leq \|\zeta_A\|_{L^1}(\delta+1)\left(\|q\|_{L^\infty} + \frac{\|p\|_{L^\infty} b^\alpha}{\Gamma(\alpha+1)}\right) \\ &\leq \delta \end{aligned}$$

Therefore $\Psi_2(U_\delta) \subset U_\delta$. Next, our aim is show the equicontinuous of $\Psi_2(U_\delta)$. Let $\eta_1, \eta_2 \in J,\ \eta_2 > \eta_1$. Then if $\varepsilon > 0$ and $\varepsilon \leq \eta_1 \leq \eta_2$, for any $y \in U_\delta$; we have

$$\begin{aligned} &|\Psi_2(y)\,(\eta_2) - \Psi_2(y)\,(\eta_1)| \\ &\leq \left|\int\limits_0^{\eta_2} S'(\eta_2 - s)\,\Psi_1(y)(\eta_2)\,ds\right| - \left|\int\limits_0^{\eta_1} S'(\eta_1 - s)\,\Psi_1(y)(\eta_1)\,ds\right| \\ &\leq \left|\int\limits_0^{\eta_1-\varepsilon} [S'(\eta_2 - s)\,\Psi_1(y)(\eta_2) - S'(\eta_1 - s)\Psi_1(y)(\eta_1)]ds\right| \\ &\quad + \left|\int\limits_{\eta_1-\varepsilon}^{\eta_1} [S'(\eta_2 - s)\,\Psi_1(y)(\eta_2) - S'(\eta_1 - s)\Psi_1(y)(\eta_1)]ds\right| \end{aligned}$$

$$+\left|\int_{\eta_1}^{\eta_2} S'(\eta_2 - s)\,\Psi_1(y)(\eta_2)\,ds\right|$$

$$\leq (\delta+1)\left(\|q\|_{L^\infty} + \frac{\|p\|_{L^\infty} b^\alpha}{\Gamma(\alpha+1)}\right)$$

$$\times \left[\begin{array}{l} \int_0^{\eta_1-\varepsilon} \left|S'(\eta_2 - s) - S'(\eta_1 - s)\right| ds + \int_{\eta_1-\varepsilon}^{\eta_1} \left|S'(\eta_2 - s) - S'(\eta_1 - s)\right| ds \\ + \int_{\eta_1}^{\eta_2} \left|S'(\eta_2 - s)\right| ds \end{array}\right]$$

as $\eta_1 \to \eta_2$ and ε is sufficiently small, the right hand side of the above inequality tends to zero. Therefore $\Psi_2(U_\delta)$ is continuous and completely continuous. Let W be a subset of U_δ such that $W \subset \overline{conv}\,(\Psi(W) \cup \{0\})$. Also W is bounded and equicontinuous, $w \to w(t) = \alpha(w(t))$ is continuous on J. By **(H5)**, Lemma (2.2) and the properties of measure α, we have for each $t \in [-r, b]$

$$\begin{aligned} w(t) &\leq \alpha\left(\Psi(W)(t) \cup \{0\}\right) \\ &\leq \alpha\left(\Psi(W)(t)\right) \\ &\leq q(t)\,\alpha(w(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s)\,\alpha(w(s))\,ds \\ &\quad + \int_0^t S'(t-s)\left(q(s)\,\alpha(w(s)) + \frac{1}{\Gamma(\alpha)}\int_0^s (s-\eta)^{\alpha-1} p(s)\,\alpha(w(\eta))\,d\eta\right) ds \\ &\leq q(t)\,w(t) + \frac{\|p\|_{L^\infty}\,\|w\|_\infty b^\alpha}{\Gamma(\alpha+1)} + \|\zeta_A\|_{L^1}\left(\|q\|_{L^\infty}\|w\|_\infty + \frac{\|p\|_{L^\infty}\|w\|_\infty b^\alpha}{\Gamma(\alpha+1)}\right). \end{aligned}$$

$$\begin{aligned} &(1-q(t))\,w(t) \\ &\quad \leq \|w\|_\infty \left(\frac{\|p\|_{L^\infty} b^\alpha\left(\|\zeta_A\|_{L^1}+1\right) + \Gamma(\alpha+1)\|\zeta_A\|_{L^1}\|q\|_{L^\infty}}{\Gamma(\alpha+1)}\right) \end{aligned}$$

This implies that

$$\|w\|_\infty\left(1 - \frac{\|p\|_{L^\infty} b^\alpha\left(\|\zeta_A\|_{L^1}+1\right) + \Gamma(\alpha+1)\|\zeta_A\|_{L^1}\|q\|_{L^\infty}}{\Gamma(\alpha+1)}\right) \leq 0.$$

By (3.1) it follows that $(1-q(t))\,w(t) = 0 \Rightarrow w(t) = 0$ for each $t \in [-r, b]$, it shows that W(t) is relatively compact. Using Arzela-Ascoli theorem, we conclude that W is relatively compact in U_δ. With reference to Theorem (3.1), Ψ has a fixed point, represents the mild solution for the system (1.1)-(1.2). ■

4. Application

Consider the following partial functional differential equation with fractional order two

$$
\begin{aligned}
&\frac{\partial^\alpha}{\partial t^\alpha}\left[u(t,\xi)-\phi_1(t)\left|u(t-\eta(u(t,\xi)),\xi)\right|^2\right]\\
&=\frac{\partial^2}{\partial x^2}u(t,\xi)+\phi_2(t)\left|\left|u(t-\eta(u(t,\xi)),\xi)\right|^2\right|\\
&\text{for } t\ge 0,\ t\in[0,b];\ \xi\in[0,\pi]\\
&u(t,0)=u(t,\pi)=0;\ t\in[0,b]\\
&u(\theta,\xi)=u_0(\theta,\xi)\quad;\quad -r\le\theta\le 0\ ;\ -\eta_{\max}\le t\le 0.
\end{aligned}
\tag{4.1}
$$

where ϕ_1 and ϕ_2 are continuous functions from [0,t] to R. The delay η is bounded, positive continuous function in R^n and assume that it is maximal delay. In order to present (4.1) in the abstract form of (1.1)-(1.2), we choose the space $E = L^2([0,\phi], R]$ and the operator $A : D(A) \subset E \to E$ is given by $Av = v''$ with domain

$$D(A) = \left\{v \in E;\ v, v' \text{ are absolutely continuous; } v'' \in E, v(0) = v(\pi) = 0\right\}.$$

Further A has a discrete spectrum with eigen values of the form $-n^2, n \in N$ whose corresponding (normalized) eigen functions are given by

$$T_n(\xi) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin(n\xi).$$

In addition, the following properties are satisfied.

i) $\{Z_n; n \in N\}$ is an orthonormal basis in E.

ii) For $u \in E, T(t)u = \sum_{n=1}^{\infty} e^{-n^2}(u, Z_n)Z_n$.

iii) For $0 < \alpha < 1, (-A)^\alpha : D(-A)^\alpha \subset E \to E$ of A is given by

$$(-A)^\alpha u = \sum_{n=1}^{\infty} n^{2\alpha}\langle u, Z_n\rangle Z_n \text{ for all } u \in D((-A)^\alpha) \text{ where}$$

$$D((-A)^\alpha) = \left\{u \in E : \sum_{n=1}^{\infty} n^{2\alpha}\langle u, Z_n\rangle Z_n \in E\right\}$$

Therefore it is clearly known that A is the infinitesimal generator of an analytic semigroup $T(t), t \ge 0$ on E and is given in (ii). It follows that $T(t), t \ge 0$ is uniformly bounded compact semigroup.

Let f and g be functions defined by

$$f(t,\zeta)\xi = \phi_1(t)|\zeta(\xi)|^2$$

and

$$g(t,\zeta)\xi = \phi_2(t)|\zeta(\xi)|^2$$

for $\zeta \in E$ and $\xi \in [0,\pi]$ are well defined continuous functions which allows to convert the system (4.1) in to the abstract system (1.1)-(1.2). Since theorem (3.2) and all the conditions are satisfied, there is a function $u \in C\left[[-r,b], L^2\right]$ which is the mild solution of (4.1).

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