Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 6 (2015), pp. 4053-4061

http://www.ripublication.com

A Novel Technique for Series Solutions to a Class of Initial Value Problems

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Abstract

In this research, we proposed a semi-analytical method to provide Taylor series solutions for the initial value problems in the form

$$\frac{\partial^2 u}{\partial t^2}(x,t) = F(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n}) + g(x)$$

with the corresponding initial conditions

$$u(x,0) = f_0(x), \qquad \frac{\partial u}{\partial t}(x,0) = f_1(x),$$

where $F(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n})$, $n \in \mathrm{N} \cup \{0\}$ is a nonlinear operator of its arguments. The methodology is noticeably forthright and easy to understand.

AMS Subject Classification: 35G25, 35C10

Keywords: Initial value problems, Nonlinear differential equations, Taylor's series solutions

1. Introduction

Nonlinear phenomena play a crucial role in designing more realistic mathematical models to describe the feasibility of nature, for example, wave propagation, traffic flow, acoustic transmission, and population dynamics. However, the treatment of those nonlinearities becomes cumbersome. The standard ways to obtain analytical solutions for nonlinear differential equations are linearization or perturbation. Although these routines should be employed to avoid the difficulties, the solutions may be changed and cannot actually represent any physical meaning. Then the

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numerical methods such as Runge-Kutta method or finite element method become more dependable in practice.

Nowadays, modern techniques have been developed to solve nonlinear differential equations in semi-analytical way, for example, the Adomian decomposition method (ADM) (see [1], [2]), the variational iteration method (VIM) (see [3], [4]), the homotopy analysis method (HAM) (see [5], [6]), and differential transform method (DTM) (see [7], [8]). The main advantage of these methods over any numerical methods is that it is free from round off errors or a massive numerical computation.

In 2011, Kutafina [9] analysed the exact solutions to scalar PDEs in the form:

$$\frac{\partial u}{\partial t}(x,t) = F(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots, \frac{\partial^n u}{\partial x^n}) + g(x).$$

The overall process is based on the ADM, but the merit of the absence of the Adomian polynomials is the highlight of this method. This modification turns into an inspiring factor for this paper.

The aim of this article is to expand the main concept of Kutafina's method to second order. The main theorem provides series solutions to initial valued problems in the Cauchy-Kovalevskaya form of second order in time. Three examples are assigned to demonstrate the usage of the theorem. The advantage of the method will be discussed in the last section.

2. Main result

Before we begin, let us introduce some notations which is used locally here for convenience. Those notations are defined with some additional details as in Table 1.

Table 1: Notations locally used in this paper

Symbol	Description	Note
$(\cdot)_{x^n}$	$\frac{\partial^n(\cdot)}{\partial x^n}$ for $n>2$	example: $u_{x^3} = \frac{\partial^3 u}{\partial x^3}$
$F[u]$	$F(u,u_x,u_{xx},\ldots,u_{x^n})$ for some n	example: $F[u]=(u,u_x,u_{xx})=uu_{xx}$
$F'[u]$	$\frac{\partial}{\partial t}F[u]$	$F'[u_0]=\left[\frac{\partial}{\partial t}F[u]\right]_{t=0}$
$F''[u]$	$\frac{\partial^2}{\partial t^2}F[u]$	$F''[u_0]=\left[\frac{\partial^2}{\partial t^2}F[u]\right]_{t=0}$
$F'''[u]$	$\frac{\partial^3}{\partial t^3}F[u]$	$F'''[u_0]=\left[\frac{\partial^3}{\partial t^3}F[u]\right]_{t=0}$
$F^{(k)}[u]$	$\frac{\partial^k}{\partial t^k}F[u]$ for $k>3$	$F^{(k)}[u_0]=\left[\frac{\partial^k}{\partial t^k}F[u]\right]_{t=0}$

Let $F(u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\ldots,\frac{\partial^n u}{\partial x^n})$ be a nonlinear operator of its arguments for some n in $\mathbb{N}\cup\{0\}$. Consider the following two-dimensional equation in the form

$$\frac{\partial^2 u}{\partial t^2}(x,t)=F(u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\ldots,\frac{\partial^n u}{\partial x^n})+g(x), \qquad (1)$$

with the corresponding initial conditions

$$u(x,0)=f_0(x), \qquad \frac{\partial u}{\partial t}(x,0)=f_1(x). \qquad (2)$$

By the Cauchy-Kovalevskaya theorem [10], there exists a unique solution u that is analytic at $(0,0)$ (note: we can always translate to $x=0$). The Taylor series solution to (1) can be obtained by the following theorem.

Theorem 1.
Let $u:\mathbb{R}\times\mathbb{R}_0^+\to\mathbb{R}$ be the unique solution of partial differential equation in the form (1), together with the initial conditions (2). Assume that u is analytic on its domain and $F[u]$ is also analytic of its arguments. Then the Taylor series for the solution $u(x,t)$ could be constructed as

$$u(x,t)=a_0+a_1t+a_2\frac{t^2}{2}+a_3\frac{t^3}{3!}+\cdots+a_n\frac{t^n}{n!}+\cdots, \qquad (3)$$

where

$$\begin{aligned}
a_0 &= f_0(x),\\
a_1 &= f_1(x),\\
a_2 &= g(x)+F[u_0],\\
a_3 &= F'[u_0],\\
&\vdots\\
a_n &= F^{(n-2)}[u_0],\\
&\vdots
\end{aligned} \qquad (4)$$

Proof. Firstly, the operator $L=\frac{\partial}{\partial t}(\cdot)$ with the inverse $L^{-1}=\int_0^t(\cdot)\,dt$ can be introduced to transform equation (1) into the following integro-differential equation:

$$u_t(x,t)=f_1(x)+g(x)t+\int_0^t F[u]\,dt. \qquad (5)$$

Apply the operator L^{-1} again to express the solution of equation (1) in the form:

$$u(x,t)=f_0(x)+f_1(x)t+g(x)\frac{t^2}{2}+\int_0^t\int_0^t F[u]\,dtdt. \qquad (6)$$

We may regard the operator $F[u]$ as an implicit function of t and then expand the

Taylor series for F about $t=0$. So we have

$$F[u]=F[u_0]+F'[u_0]t+F''[u_0]\frac{t^2}{2}+F'''[u_0]\frac{t^3}{3!}+\cdots+F^{(n)}[u_0]\frac{t^n}{n!}+\cdots, \quad (7)$$

where the coefficients are acquired by the chain rule, for example, $F'[u]$ can be computed by $\sum_{i=0}^{n-1}F_{u_{x^i}}[u]\cdot u_{x^i t}$, where u_{x^0} means u itself. Substitute (7) back into (6) to reach

$$u(x,t)=f_0(x)+f_1(x)t+\left[g(x)+F[u_0]\right]\frac{t^2}{2}+F'[u_0]\frac{t^3}{3!}+\cdots+F^{(n-2)}[u_0]\frac{t^n}{n!}+\cdots. \quad (8)$$

Therefore, we can express (8) in the form of (3)

$$u(x,t)=a_0+a_1t+a_2\frac{t^2}{2}+a_3\frac{t^3}{3!}+\cdots+a_n\frac{t^n}{n!}+\cdots,$$

where $a_0,a_1,a_2,\ldots$ are explicitly defined in (4) as claimed. □

Remark:
By the construction of the method, we automatically have that

$$\begin{aligned} u(x,0)&=a_0,\\ u_t(x,0)&=a_1,\\ u_{t^2}(x,0)&=a_2,\\ u_{t^3}(x,0)&=a_3,\\ &\vdots\\ u_{t^n}(x,0)&=a_n,\\ &\vdots \end{aligned} \quad (9)$$

Once $a_0,a_1,a_2,\ldots,a_n$ for some small n are computed, an approximate solution to (1) can be accomplished as a truncated series of (3).

3. Illustrative examples

Three difference examples are considered in this section to illustrate the effectiveness of the new method described in the previous part.We first consider the linear wave equation [8] in Example 3.1 in order to clearly illustrate the usage of Theorem 1. Example 3.2 focuses on the potential of the method on the treatment of nonlinearity in the Sine-Gordon equation [11]. The last example shows the approximate solution to the nonlinear Klein-Gordon equation [12] which is in the form of truncated series obtained from our approach. The efficiency of the method is assessed by comparison with the exact solutions.

Example 3.1
Consider the linear wave equation

$$u_{tt} = u_{xx} - 3u, \tag{10}$$

with the initial conditions:

$$u(x,0) = 0, \qquad u_t(x,0) = 2\cos x. \tag{11}$$

Solution.
In this homogeneous problem, we have

$$a_0 = f_0(x) = 0,$$
$$a_1 = f_1(x) = 2\cos x,$$
$$\text{and } F[u] = F(u, u_x, u_{xx}) = u_{xx} - 3u.$$

The first derivatives of F with respect to its arguments are performed as:

$$F_u = -3, \quad F_{u_x} = 0, \quad F_{u_{xx}} = 1,$$

while the derivatives of higher order are all zero. By applying Theorem 1, we can directly calculate the Taylor coefficients as follows:

$$\begin{aligned} a_0 &= 0, & a_1 &= 2\cos x, \\ a_2 &= 0, & a_3 &= F_u[a_0]a_1 + F_{u_{xx}}[a_0](a_1)_{xx} \\ & & &= -8\cos x, \\ a_4 &= 0, & a_5 &= F_u[a_0]a_3 + F_{u_{xx}}[a_0](a_3)_{xx} \\ & & &= 32\cos x, \\ &\vdots & &\vdots \end{aligned} \tag{12}$$

So, we conclude that

$$\begin{aligned} u(x,t) &= (2\cos x)t - (8\cos x)\frac{t^3}{3!} + (32\cos x)\frac{t^5}{5!} - \cdots \\ &= \sum_{n=1}^{\infty} (-1)^{n-1} \cos x \frac{(2t)^{2n-1}}{(2n-1)!} \end{aligned} \tag{13}$$

is the semi-analytic solution of (10) and (11), which matches to the Taylor series of the exact solution $u(x,t) = \cos x \sin 2t$ around $t = 0$. □

Example 3.2
Consider the Sine-Gordon equation

$$u_{tt} = u_{xx} - \sin u, \tag{14}$$

with the initial conditions:

$$u(x,0) = 0, \qquad u_t(x,0) = 4\operatorname{sech} x. \tag{15}$$

Solution.
We first set

$$F[u] = F(u, u_x, u_{xx}) = u_{xx} - \sin u.$$

Then we collect all nonzero partial derivatives of F here:

$$F_u = -\cos u, \quad F_{u_{xx}} = 1, \quad F_{uu} = \sin u, \quad F_{uuu} = \cos u.$$

Implementing the aforesaid method, we successively have the following expressions

$$\begin{aligned}
a_0 &= 0, \quad a_1 = 4\operatorname{sech} x, \\
a_2 &= 0, \quad a_3 = F_u[a_0]a_1 + F_{u_{xx}}[a_0](a_1)_{xx} \\
&\qquad\quad = -8\operatorname{sech}^3 x, \\
a_4 &= 0, \quad a_5 = F_u[a_0]a_3 + F_{u_{xx}}[a_0](a_3)_{xx} \\
&\qquad\quad + F_{uu}[a_0](3a_2a_1) + F_{uuu}(a_1)^3 \\
&\qquad\quad = 96\operatorname{sech}^5 x, \\
&\vdots \qquad\quad \vdots
\end{aligned} \tag{16}$$

The series solution is then given by

$$\begin{aligned}
u(x,t) &= (4\operatorname{sech} x)t - \left(\frac{4\operatorname{sech}^3 x}{3}\right)\frac{t^3}{3!} + \left(\frac{4\operatorname{sech}^5 x}{5}\right)\frac{t^5}{5!} - \cdots \\
&= 4\sum_{n=1}^{\infty}(-1)^{n-1}\operatorname{sech}^{2n-1} x \frac{(t)^{2n-1}}{(2n-1)!},
\end{aligned} \tag{17}$$

which can be expressed in closed form as $u(x,t) = 4\arctan(t\sec x)$. □

Example 3.3

Consider the nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + \frac{3}{4}u - \frac{3}{2}u^3 = 0, \tag{18}$$

with the initial conditions:

$$u(x,0) = -\operatorname{sech} x, \qquad u_t(x,0) = \frac{1}{2}\operatorname{sech} x \tanh x. \tag{19}$$

Solution.

By means of the proposed method, we set

$$F[u] = F(u, u_x, u_{xx}) = u_{xx} - \frac{3}{4}u + \frac{3}{2}u^3$$

and the nonzero derivatives of F are calculated as follows:

$$F_u = -\frac{3}{4} + \frac{9}{2}u^2, \quad F_{u_{xx}} = 1, \quad F_{uu} = 9u, \quad F_{uuu} = 9.$$

From the initial conditions, we already have

$$a_0 = -\operatorname{sech} x, \quad \text{and} \quad a_1 = \frac{1}{2}\operatorname{sech} x \tanh x.$$

We continue computing the coefficients a_2, a_3 and a_4 according to the formulae (4). Here are the results:

$$\begin{aligned} a_2 &= F[a_0] = \frac{\partial^2 a_0}{\partial x^2} - \frac{3}{4}a_0 + \frac{3}{2}{a_0}^3, \\ &= \frac{1}{4}\operatorname{sech} x(1-2\tanh^2 x), \\ a_3 &= F_u[a_0]a_1 + F_{u_{xx}}[a_0](a_1)_{xx} \\ &= \frac{1}{8}\tanh x \operatorname{sech} x(6\tanh^2 x - 5), \\ a_4 &= F_u[a_0]a_2 + F_{u_{xx}}[a_0](a_2)_{xx} + F_{uu}[a_0](a_1)^2 \\ &= -\frac{1}{16}\operatorname{sech}^5 x(\cosh^4 x - 20\cosh^2 x + 24). \end{aligned} \tag{20}$$

Let $\tilde{u}_k = \sum_{n=0}^{k} a_n \frac{t^n}{n!}$. It seems hard to find the correct pattern of $\tilde{u}_k$. However, by the assumption of Theorem 1 that the unique solution is analytic at $(0,0)$, the sequence $\{\tilde{u}_k\}_{k=0}^{\infty}$ converges to the exact solution for t near 0. As a consequence, the solution of this problem is given by

$$u(x,t) = \lim_{k\to\infty} \tilde{u}_k.$$

The truncated series can be used to compute function values numerically. Figure 1 shows the comparisons between the 4-term approximation $\tilde{u}_4$ (a) and the exact solution $u(x,t) = -\operatorname{sech}(x + t/2)$ (b) for the region $[-2\pi, 2\pi] \times [0,2]$. From Figure 1, we can see that the 4-term approximation coincides with the exact solution with a very small fluctuation at the terminal time. □

4. Conclusion

The main concern of this article is to provide series solutions to initial valued problems in the Cauchy-Kovalevskaya form of second order in time.We have achieved this goal by extending the main concept of Kutafina's method. By a morally forthright algebraic computation, we easily obtain a set of certain formulae of the components of the Taylor series solutions. The first two examples show that the resultant theorem reported in section 2 is applicable to both linear and nonlinear operators. The last example indicates that only a small size of computation can achieve a good approximation to the solution. Finally, it is remarked that this method can be extended further for more complex models.

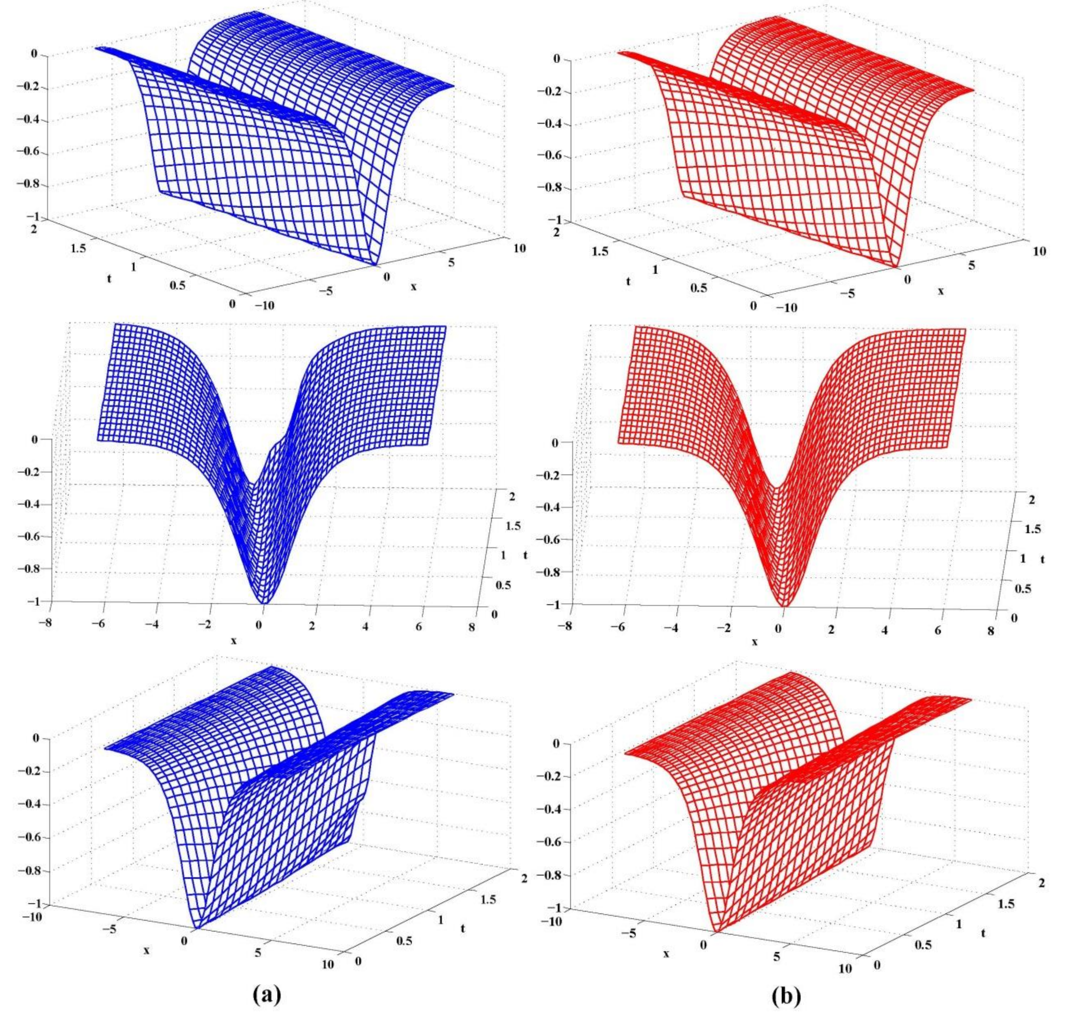

Figure 1: (a): The numerical results for $\tilde{u}_4$ in Example 3.3 in comparison with (b): the analytic solution $u(x, t) = -\operatorname{sech}(x + t/2)$.

Acknowledgement

The authors are grateful to the reviewers for valuable suggestions in improving the quality of the paper. This study is supported by Faculty of Science, Burapha University.

References

[1] Adomian, G., 1988, "A review of the decomposition method in applied Mathematics, " J. Math. Anal. Appl., 135, pp. 501-544.

[2] Wazwaz, A. M., 2000, "A new algorithm for calculation Adomian polynomials for nonlinear operators", Appl. Math. Comput., 111, pp. 33-51.

[3] He J. H., 1999, “Variational iteration method - A kind of non-linear analytical technique: Some examples, ” Int. J. Non-Linear Mech., 34, pp. 699-708.
[4] He J. H., 2007, “Variational iteration method - Some recent results and new interpretations”, J. Comput. Appl. Math., 207, pp. 3-17.
[5] Liao, S. J., 2004, “On the homotopy analysis method for nonlinear problems”, Appl. Math. Comput., 147, pp. 499-513.
[6] Liao, S. J., 2009, “Notes on the homotopy analysis method: Some definitions and theorems”, Commun. Nonlinear Sci. Numer.Simul., 17(4), pp. 983-997.
[7] Abdel-Halim Hassan, I. H., 2004, “Differential transformation technique for solving higher-order initial value problems”, Appl. Math. Comput., 154, pp. 299-311.
[8] Keskin, Y., Oturanc, G., 2009, “Reduced differential transform method for partial differential equations”, Int. J. Nonlinear Sci. Numer. Simul., 10(6), pp. 741-749.
[9] Kutafina, E., 1994, “Taylor Series for Adomian Decomposition Method”, Int. J. Comput. Math., 88(17), pp. 3677-3684.
[10] Hozegel, G., 2014 (March), “Partial Differential Equations (Week 3+4) Cauchy-Kovalevskaya and Holmgren”, Imperial College London.
[11] Keskin, Y., Caglar, I., and Koc, A. B., 2011, “Numerical Solution of Sine-Gordon Equation by Reduced Differential Transform Method”, In: Proceedings of the World Congress on Engineering 2011 Vol I, London, pp. 111-115.
[12] Keskin, Y., Servi, S., and Oturanc, G., 2011, “Reduced Differential Transform Method for Solving Klein Gordon Equations”, In: Proceedings of the World Congress on Engineering 2011 Vol I, London, pp. 267-271.