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Equitable Independent Saturation Number of a Graph

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Abstract

Let G=(V,E) be a simple graph. A subset D of V(G) is said to be an equitable dominating set of G if for every vertex $v \in V-D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d(u)-d(v)| \leq 1$. Let $v \in V(G)$. Define $IS^e(v)=\max\{|S|: S$ is an equitable independent set of G containing $v\}$. Define $IS^e(G)=\min\{IS^e(v): v \in V(G)\}$. $IS^e(G)$ is called the equitable independent saturation number of G. The values of $IS^e(G)$ for many classes of graphs have been found. Interesting results are proved with respect to the new parameters.

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Definition 1.1A subset S of V(G) is said to be degree equitable if $|d_G(u)-d_G(v)| \leq 1$, for all $u,v \in S$.
S is called an equitable dominating set of G if for every $u \in V-S$, there exists $v \in S$ such that $uv \in E(G)$ and $|d_G(u)-d_G(v)| \leq 1$. This domination parameter was studied in [6].

Definition 1.2 (Equitable Independent Sets) [6]
Let $u \in V(G)$. The equitable neighbourhood of u denoted by $N^e(u)$ is defined as $N^e(u)=\{v \in V: v \in N(u)$ and $|d(u)-d(v)| \leq 1\}$. u is called an equitable isolate if $N^e(u)=\varphi$. The set of all equitable isolates of G is denoted by I_e. The equitable degree of u denoted by $d_G{}^e(u)$ is $|N^e(u)|$.

Definition 1.3 [6] A subset S of V(G) is called an equitable independent set if for any $u \in S$, $v \notin N^e(u)$ for all $v \in S-\{u\}$.

Definition 1.4 The minimum cardinality of a maximal equitable independent set of G is denoted by $i^e(G)$ and the maximum cardinality of a maximal equitable independent set of G is called the equitable independence number of G and is denoted by $\beta_0^e(G)$.

Introduction 1.5 Given a graph G the usual domination gives rise to an inequality chain namely $er(G) \leq ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G) \leq ER(G)$. This chain is constructed using maximality and minimality conditions [5]. S. Arumugam and [3,1,2,4] have defined independent saturation and irredundant saturation numbers and extended the inequality chain. Domination saturation number was conceived by B.D. Acharya. These saturation parameters are defined in the case of Equitable domination.

Definition 1.6 Let $v \in V(G)$. Define $IS^e(v)=\max\{|S|: S$ is an equitable independent set of G containing $v\}$. Define $IS^e(G)=\min\{IS^e(v): v \in V(G)\}$. $IS^e(G)$ is called the equitable independent saturation number of G.

Remark 1.7 $i^e(G) \leq IS^e(G) \leq \beta_0^e(G)$

For:Let S be an IS^e-set of a vertex u in G. Then S is a maximal equitable independent set of G containing u. Therefore, $i^e(G) \leq IS^e(u)$ for every u in V(G). Therefore, $i^e(G) \leq \min\{IS^e(u): u \in V(G)\} = IS^e(G)$.

Remark 1.8 $\gamma^e(G) \leq i^e(G) \leq IS^e(G) \leq \beta_0^e(G) \leq \Gamma^e(G)$

$IS^e(G)$ for Standard Graphs

$(i)\ IS^e(K_n) = 1$

$(ii) IS^e(\overline{K_n}) = n$

$(iii)\ IS^e(K_{1,n}) = n + 1$

$(iv)\ IS^e(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$

$(v)\ IS^e(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$

$(vi)\ IS^e(W_n) = 1$

$(vii)\ IS^e(P) = 3$, where P is the Petersen graph.

$(viii)\ IS^e(K_{m,n}) = min\{m, n\}$

$$(ix)\ IS^e(D_{r,s}) = \begin{cases} r+s+1, if\ |r-s| \geq 2 \\ r+s+1, if\ |r-s| = 1, r,s \geq 2 \\ r+s+1, if\ |r-s| \geq 2\ and\ either\ r\ or\ s = 1 \\ r+s, if\ r=1, s=2 \end{cases}$$

Remark 1.9 Let G be any graph. Let $v \in V(G)$. $N^e(v)$ contains equitable neighbours of v. Any equitable independent set of G containing v cannot contain the vertices of $N^e(v)$. Therefore, if S is an equitable independent set of G containing v, then $S \subseteq V-$

$N^e(v)$. $|S|\leq n-|N^e(v)|=n-deg_G{}^e(v)$. Therefore, $IS^e(v)\leq n-deg_G{}^e(v)$. Therefore, $IS^e(G)\leq\min\{n-deg_G{}^e(v): v\in V(G)\}$.

Remark 1.10 Let G be a complete multi partite graph with t partite sets in which every partite set has the same cardinality say s. $IS^e(v)=s$ for every $v\in V(G)$. Therefore, $\sum_{v\in V(G)} IS^e(v) = ts^2$.
$\sum_{v\in V(G)} deg(v) = (t-1)s(ts) = t^2s^2 - ts^2$.
$2q = t^2s^2 - ts^2 = t^2s^2 - \sum_{v\in V(G)} IS^e(v)$. $\sum_{v\in V(G)} IS^e(v) = t^2s^2 - 2q = p^2 - 2q$.

Remark 1.11 There are multi partite graphs in which $\sum_{v\in V(G)} IS^e(v) \neq p^2 - 2q$.
For: Consider $K_{5,3,5}$. Let V_1,V_2,V_3 be the partite sets with cardinality 5,3 and 5. For any vertex v in V_1, $IS^e(v)=8$. Also, $IS^e(v)=13$ for any v in V_2 and $IS^e(v)=8$ for any v in V_3. $\sum_{v\in V(G)} IS^e(v) = 5*8 + 3*10 + 5*8 = 110$.p=total number of vertices=13. q=(sum of degrees of v)/2=55. p^2-2q=169-110=59. Therefore, $\sum_{v\in V(G)} IS^e(v) \neq p^2 - 2q$.

Theorem 1.12 Let G be a complete multi partite graph in which the partite sets are equitable. Then

$$\sum_{v\in V(G)} IS^e(v) = p^2 - 2q.$$

Proof:
Let $V_1,V_2,\ldots,V_k$ be the partite sets of G. Let $|V_i|=t_i$, $1\leq i\leq k$. Let $t_1,t_2,\ldots,t_k$ be equitable. $IS^e(v)=t_i$, if $v\in V_i$, $1\leq i\leq k$.Therefore, $\sum_{v\in V(G)} IS^e(v) = \sum_{i=1}^{k} t_i{}^2$.
2q=sum of the degrees of the vertices

$$= t_1(t_2 + t_3 + \cdots + t_k) + t_2(t_1 + t_3 + \cdots + t_k) + \cdots + t_k(t_1 + t_2 + \cdots + t_{k-1})$$
$$= t_1(\textstyle\sum_{i=1}^{k} t_i) + t_2(\sum_{i=1}^{k} t_i) + \cdots + t_k(\sum_{i=1}^{k} t_i) - t_1{}^2 - t_2{}^2 - \cdots - t_k{}^2$$
$$= t_1p + t_2p + \cdots + t_kp - t_1{}^2 - t_2{}^2 - \cdots - t_k{}^2$$
$$= p^2 - t_1{}^2 - t_2{}^2 - \cdots - t_k{}^2$$

$p^2 - 2q = \sum_{i=1}^{k} t_i{}^2$. Therefore, $\sum_{v\in V(G)} IS^e(v) = p^2 - 2q$.

Theorem 1.13 Let G be a complete multi partite graph in which at least two partite sets are not equitable. Then $\sum_{v\in V(G)} IS^e(v) \neq p^2 - 2q$.

Proof:
Let $V_1,V_2,\ldots,V_k$ be the partite sets of G. Let $|V_i|=t_i$, $1\leq i\leq k$. Suppose $|V_1|$ and $|V_2|$ are not equitable. Then $IS^e(v) \geq t_1 + t_2$ for every $v \in V_1 \cup V_2$.

$$\sum_{v\in V(G)} IS^e(v) \geq t_1(t_1 + t_2) + t_2(t_1 + t_2) + t_3{}^2 + \cdots + t_k{}^2$$
$$= t_1{}^2 + t_2{}^2 + \cdots + t_k{}^2 + 2t_1t_2$$

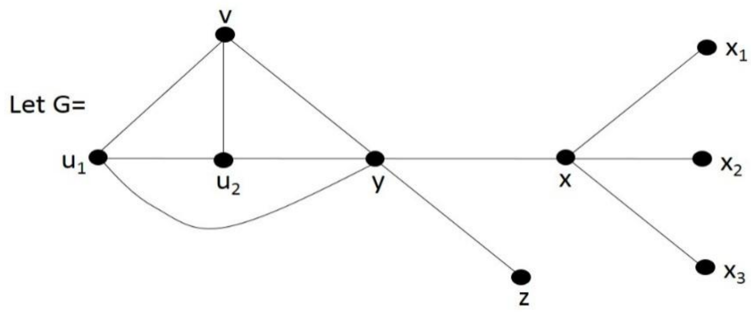
$$p^2 - 2q = (t_1 + t_2 + \cdots + t_k)^2 - t_1(t_2 + t_3 + \cdots + t_k) - t_2(t_1 + t_3 + \cdots + t_k) - \cdots - t_k(t_1 + t_2 + \cdots + t_{k-1})$$
$$= (t_1 + t_2 + \cdots + t_k)^2 - (t_1 + t_2 + \cdots + t_k)^2 + {t_1}^2 + {t_2}^2 + \cdots + {t_k}^2$$
$$= {t_1}^2 + {t_2}^2 + \cdots + {t_k}^2$$
$$< \sum_{v\in V(G)} IS^e(v)$$

Therefore, $\sum_{v\in V(G)} IS^e(v) \neq p^2 - 2q$.

Remark 1.14 Let G be a complete multi partite graph with $\sum_{v\in V(G)} IS^e(v) = p^2 - 2q$. Then any two partite sets are equitable.

Remark 1.15 Let G be a complete multi partite graph with $\sum_{v\in V(G)} IS^e(v) = p^2 - 2q$ if and only if every two partite sets are equitable.

Illustration 1.16



$N[v]=\{v,u_1,u_2,y\}$, $N^e(v)=\{u_1,u_2\}$, $N[v]-N^e(v)=\{v,y\}$, $V-N[v]=\{z,x,x_1,x_2,x_3\}$,
β_0^e-set of $V-N[v] =\{z,x,x_1,x_2,x_3\}$, $N[v]-N^e(v)\cup \{z,x,x_1,x_2,x_3\}=\{v,y,z,x,x_1,x_2,x_3\}$
This set is not equitable independent since x and y are adjacent and equitable. Therefore,
$IS^e(v)<|N[v]-N^e(v)|+\beta_0^e(V-N[v])$.

Theorem 1.17 Let G be a graph and $v\in V(G)$. Then $IS^e(v)=|N[v]-N^e(v)|+|T|$ where T is a maximum equitable independent subset of $V-N[v]$ such that no vertex of T is equitably adjacent with any vertex of $N[v]-N^e(v)$.

Proof:

Let S be an $IS^e(v)$-set of v. Any equitable neighbour of v cannot belong to S since $v\in S$. Therefore, $N[v]-N^e(v)\subseteq S$. Let $x\in\beta_0^e(V-N[v])$. Therefore, $x\notin N[v]$ and x is equitably independent in $V-N[v]$. If x is not equitably adjacent with any vertex of $N[v]-N^e(v)$ then $x\in S$. Let T be a subset of $V-N[v]$ such that T is maximum equitably independent in $V-N[v]$, with the property that no element of T is equitable with any vertex of $N[v]-N^e(v)$. Therefore, $(N[v]-N^e(v))\cup T \subseteq S$. Let $y\in V$. If $y\in N[v]-N^e(v)$ then $y\in S$. If $y\notin N[v]-N^e(v)$ then either y is not adjacent to v or $y\in N[v]\cap N^e(v)$. That is, $y\notin N[v]$ or $y\in N^e(v)$. If $y\in N^e(v)$ then $y\notin S$. Let $y\in V-N[v]$. If y is equitably independent in $V-N[v]$ then $y\in S$ provided y is not equitable with any vertex of $N[v]-N^e(v)$. Therefore, $(N[v]-N^e(v))\cup T \subseteq S$. Let $z\in S$. Either z is adjacent with v or not adjacent with v. If z is adjacent with v then $z\in N[v]-N^e(v)$. If z is not adjacent with v then z is equitably independent in $V-N[v]$ and z is not equitable with any vertex of $N[v]-N^e(v)$ where degree of any vertex α in $V-N[v]$ is degree of α in G. Therefore, $|S|\subseteq |(N[v]-N^e(v))\cup T|$ where T is a maximum equitable independent set of $V-N[v]$ such that no vertex of T is equitably adjacent with any vertex of $N[v]-N^e(v)$. Therefore, $|S|=|N[v]-N^e(v)|+|T|$.

Remark 1.18 Let G be a graph. Let $v\in V(G)$. Then $IS^e(v)$ may be strictly greater than $|N[v]-N^e(v)|+\beta_0(V-N[v])$.

Remark 1.19 $IS(v)=\beta_0(V-N(v))$. $IS^e(v)=|N[v]-N^e(v)|+|T|$ where T is a maximum equitable independent subset of $V-N[v]$ such that no vertex of T is equitably adjacent with any vertex of $N[v]-N^e(v)$. $IS^e(v)-IS(v) =|N[v]-N^e(v)|+|T|-\beta_0(V-N(v))$.

Theorem 1.20 Let G be a graph. $IS^e(G)=1$ if and only if there exists a vertex $v\in V(G)$ such that $deg_G{}^e(v)=n-1$.

Proof:
Suppose $deg_G{}^e(v)=n-1$. Then $deg_G(v)=n-1$. $IS^e(v)=|N[v]-N^e(v)|+|T|=1+0$. Therefore, $IS^e(G)=1$ (since $T=\varphi$). Conversely, Let $IS^e(G)=1$. Therefore, there exists a vertex v in V(G) such that $IS^e(v)=1$. That is,
$|N[v]-N^e(v)|+|T|=1$. Since $|N[v]-N^e(v)| \geq 1$, we get that $|N[v]-N^e(v)|=1$ and $|T|=0$. Therefore, any vertex of $V-N[v]$ is equitably adjacent with some vertex of $N[v]-N^e(v)=\{v\}$. Therefore, every vertex other than v is equitably adjacent with v. Therefore, $deg^e{}_G(v)=n-1$.

Theorem 1.21 Let G be a graph. $IS^e(G)=2$ if $\Delta^e(G) \leq n-2$ and either there exists a vertex $v\in V(G)$ with $deg_G(v)=n-1$ and $deg_G{}^e(v)=n-2$ or there is a vertex $v\in V(G)$ such that $deg_G(v)=deg_G{}^e(v)=n-2$.

Proof:
From theorem, if $IS^e(v)=1$ then $deg_G{}^e(v)=n-1$. Since $\Delta^e(G) \leq n-2$, $IS^e(v)>1$ for any $v\in V(G)$. Therefore, $IS^e(G) \geq 2$. Suppose there exists a vertex v with $deg_G(v)=n-1$ and $deg_G{}^e(v)=n-2$. $IS^e(v)=|N[v]-N^e(v)|+|T|=2+0$ (since $deg_G(v)=n-1$ implies $V=N[v]$). Therefore, $IS^e(G) \leq 2$. But $IS^e(G) \geq 2$. Therefore, $IS^e(G)=2$. Suppose there exists a vertex v such that $deg_G(v)=deg_G{}^e(v)=n-2$. $IS^e(v)=1+1=2$. Therefore, $IS^e(G) \leq 2$. But $IS^e(G) \geq 2$. Therefore, $IS^e(G)=2$.

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