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Symmetric identities for Carlitz's generalized twisted (h, q)-Bernoulli polynomials associated with p-adic q-integral on $\mathbb{Z}_p$

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Abstract

In this paper, we study the symmetry for the generalized twisted (h, q)-Bernoulli numbers $\beta^{(h)}_{n,\chi,q,\zeta}$ and polynomials $\beta^{(h)}_{n,\chi,q,\zeta}(x)$. We obtain some interesting identities of the power sums and the generalized twisted polynomials $\beta^{(h)}_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant q-integral on $\mathbb{Z}_p$.

AMS subject classification: 11B68, 11S40, 11S80.
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1. Introduction

Many mathematicians have studies different kinds of the Euler, Bernoulli, Genocchi, tangent numbers and polynomials (see [1-7]). These numbers and polynomials play important roles in many different areas of mathematics such as number theory, combinatorics, special function and analysis. Recently, Y. Hu studied several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field (see [1]). D. Kim *et al.* [3] derived some identities of symmetry for generalized Carlitz's q-Bernoulli numbers and polynomials by using the p-adic q-integrals on $\mathbb{Z}_p$ in p-adic field. The purpose of this paper is to obtain some interesting identities of the power sums and generalized twisted (h, q)-Bernoulli polynomials $\beta^{(h)}_{n,\chi,q,\zeta}(x)$ using the symmetric properties for the p-adic q-invariant integral on $\mathbb{Z}_p$.

Let p be a fixed prime number. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [1-4])} .$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. Let

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}.$$

For $g \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(g) = \int_{\mathbb{Z}_p} g(x)d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} g(x)q^x \text{ (cf. [3-6])} . \tag{1.1}$$

Let a fixed positive integer d with $(p, d) = 1$, set

$$X = X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. It is easy to see that

$$\int_X g(x)d\mu_q(x) = \int_{\mathbb{Z}_p} g(x)d\mu_q(x), \text{ (see [2]).} \tag{1.2}$$

We assume that $h \in \mathbb{Z}$. Let $T_p = \cup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\zeta | w^{p^N} = 1\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$ (see [7]).

2. Symmetric identities for Carlitz's generalized twisted (h, q)-Bernoulli numbers and polynomials

D. Kim *et al.* [3] investigated interesting properties of symmetry p-adic invariant q-integral on $\mathbb{Z}_p$ for generalized q-Bernoulli polynomials. By using same method of [3], expect for obvious modifications, we obtain some symmetric properties for generalized twisted (h, q)-Bernoulli polynomials. If we take $\zeta = 1$ in all equations of this article, then [3] are the special case of our results. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $(d, p) = 1$. For $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$, the twisted (h, q)-Bernoulli polynomials $\beta^{(h)}_{n,q,\zeta}(x)$ are defined by

$$\beta^{(h)}_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y)q^{(h-1)y}[x + y]^n_q d\mu_q(y).$$

When $x=0$, $\beta^{(h)}_{n,q,\zeta}=\beta^{(h)}_{n,q,\zeta}(0)$ is called the n-th twisted (h,q)-Bernoulli numbers $\beta^{(h)}_{n,q,\zeta}$. We introduce the generalized twisted (h,q)-Bernoulli polynomials $\beta^{(h)}_{n,\chi,q,\zeta}(x)$ attached to χ. The generalized twisted (h,q)-Bernoulli polynomials $\beta^{(h)}_{n,\chi,q,\zeta}(x)$ attached to χ are defined by

$$\beta^{(h)}_{n,\chi,q,\zeta}(x)=\int_X \chi(y)\phi_\zeta(y)q^{(h-1)y}[x+y]_q^n d\mu_q(y).$$

When $x=0$, $\beta^{(h)}_{n,\chi,q,\zeta}=\beta^{(h)}_{n,\chi,q,\zeta}(0)$ is called the n-th generalized twisted (h,q)-Bernoulli numbers $\beta^{(h)}_{n,\chi,q,\zeta}$. By using p-adic q-integral, we obtain

$$\beta^{(h)}_{n,\chi,q,\zeta}(x)=[d]_q^{n-1}\sum_{i=0}^{d-1}\chi(i)q^{hi}\zeta^i\beta^{(h)}_{n,\chi,q^d,\zeta^d}(\frac{x+a}{d}).$$

We note that

$$\sum_{n=0}^{\infty}\beta^{(h)}_{n,\chi,q,\zeta}\frac{t^n}{n!}=\int_X \chi(y)\zeta^y q^{(h-1)y}e^{[x+y]_q t}d\mu_q(x).$$

Let w_1 and w_2 be natural numbers. Then, by (1.2), we obtain

$$\begin{aligned}
&\frac{1}{[w_1]_q}\int_X \chi(y)\zeta^{w_1 y}q^{w_1(h-1)y}e^{[w_1w_2x+w_2j+w_1y]_q t}d\mu_{q^{w_1}}(y)\\
&=\lim_{N\to\infty}\frac{1}{[w_1]_q}\frac{1}{[dw_2p^N]_{q^{w_1}}}\sum_{y=0}^{dw_2p^N-1}\chi(y)\zeta^{w_1y}q^{w_1(h-1)y}e^{[w_1w_2x+w_2j+w_1y]_q t}q^{w_1y}\\
&=\lim_{N\to\infty}\frac{1}{[dw_1w_2p^N]_q}\sum_{i=0}^{dw_2-1}\chi(i)q^{w_1hi}\zeta^{w_1i}\sum_{y=0}^{p^N-1}\zeta^{dw_1w_2y}q^{dw_1w_2hy}\\
&\qquad\times e^{[w_1w_2x+w_2j+w_1i+dw_1w_2y]_q t}
\end{aligned}\tag{2.1}$$

From (2.1), we can derive the following equation (2.2):

$$\begin{aligned}
&\frac{1}{[w_1]_q}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2j}q^{hw_2j}\int_X \chi(y)\zeta^{w_1y}q^{w_1(h-1)y}e^{[w_1w_2x+w_2j+w_1y]_q t}d\mu_{q^{w_1}}(y)\\
&=\lim_{N\to\infty}\frac{1}{[dw_1w_2p^N]_q}\sum_{j=0}^{dw_1-1}\sum_{i=0}^{dw_2-1}\sum_{y=0}^{p^N-1}\chi(i)\chi(j)\zeta^{w_2j}\zeta^{w_1i}q^{w_2hj}q^{w_1hi}\\
&\qquad\times e^{[w_1w_2x+w_2j+w_1i+dw_1w_2y)]_q t}\zeta^{dw_1w_2y}q^{dw_1w_2hy}
\end{aligned}\tag{2.2}$$

By the same method as (2.2), we obtain

$$\begin{aligned}&\frac{1}{[w_2]_q}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1 j}q^{hw_1 j}\int_X \chi(y)\zeta^{w_2 y}q^{w_2(h-1)y}e^{[w_1w_2x+w_1j+w_2y]_q t}d\mu_{q^{w_2}}(y)\\&=\lim_{N\to\infty}\frac{1}{[dw_1w_2p^N]_q}\sum_{j=0}^{dw_2-1}\sum_{i=0}^{dw_1-1}\sum_{y=0}^{p^N-1}\chi(i)\chi(j)\zeta^{w_1 i}\zeta^{w_2 i}q^{w_1hj}q^{w_2hi}\\&\qquad\times e^{[w_1w_2x+w_1j+w_2i+dw_1w_2y]_q t}\zeta^{dw_1w_2y}q^{dw_1w_2hy}\end{aligned}\tag{2.3}$$

Therefore, by (2.2) and (2.3), we have the following theorem.

Theorem 2.1. For $w_1, w_2 \in \mathbb{N}$, we have

$$\begin{aligned}&\frac{1}{[w_1]_q}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2 j}q^{hw_2 j}\int_X \chi(y)\zeta^{w_1 y}q^{w_1(h-1)y}e^{[w_1w_2x+w_2j+w_1y]_q t}d\mu_{q^{w_1}}(y)\\&=\frac{1}{[w_2]_q}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1 j}q^{hw_1 j}\int_X \chi(y)\zeta^{w_2 y}q^{w_2(h-1)y}e^{[w_1w_2x+w_1j+w_2y]_q t}d\mu_{q^{w_2}}(y).\end{aligned}\tag{2.4}$$

By substituting Taylor series of e^{xt} into (2.4) and after elementary calculations, we obtain the following corollary.

Corollary 2.2. For $w_1, w_2 \in \mathbb{N}$, $n \geq 0$, we have

$$\begin{aligned}&[w_1]_q^{n-1}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2 j}q^{hw_2 j}\int_X \chi(y)\zeta^{w_1 y}q^{w_1(h-1)y}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_{q^{w_1}}(y)\\&=[w_2]_q^{n-1}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1 j}q^{hw_1 j}\int_X \chi(y)\zeta^{w_2 y}q^{w_2(h-1)y}\left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}^n d\mu_{q^{w_2}}(y).\end{aligned}$$

By Corollary 2.2, we have the following theorem.

Theorem 2.3. For $w_1, w_2 \in \mathbb{N}$, $n \geq 0$, we have

$$\begin{aligned}&[w_1]_q^{n-1}\sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2 j}q^{hw_2 j}\beta^{(h)}_{n,\chi,q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\&=[w_2]_q^{n-1}\sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1 j}q^{hw_1 j}\beta^{(h)}_{n,\chi,q^{w_2},\zeta^{w_2}}\left(w_1x+\frac{w_1}{w_2}j\right).\end{aligned}$$

By Theorem 2.3, we can derive the following equation (2.5):

$$\begin{aligned}
&\int_X \chi(y)\zeta^{w_1 y} q^{w_1(h-1)y}\left[w_2 x + \frac{w_2}{w_1} j + y\right]^n_{q^{w_1}} d\mu_{q^{w_1}}(y)\\
&= \sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i [j]^i_{q^{w_2}} q^{w_2(n-i)j}\int_X \chi(y)\zeta^{w_1 y} q^{w_1(h-1)y}\left[w_2 x + y\right]^{n-i}_{q^{w_1}} d\mu_{q^{w_1}}(y)\\
&= \sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i [j]^i_{q^{w_2}} q^{w_2(n-i)j}\beta^{(h)}_{n-i,\chi,q^{w_1},\zeta^{w_1}}(w_2 x)\,.
\end{aligned} \tag{2.5}$$

By (2.5), and Theorem 2.3, we have

$$\begin{aligned}
&[w_1]^{n-1}_q \sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2 j} q^{hw_2 j}\int_X \chi(y)\zeta^{w_1 y} q^{w_1(h-1)y}\left[w_2 x + \frac{w_2}{w_1} j + y\right]^n_{q^{w_1}} d\mu_{q^{w_1}}(y)\\
&= \sum_{j=0}^{dw_1-1}\chi(j)\zeta^{w_2 j} q^{hw_2 j}\sum_{i=0}^{n}\binom{n}{i}[w_2]^i_q[w_1]^{n-i-1}_q[j]^i_{q^{w_2}} q^{w_2(n-i)j}\beta^{(h)}_{n-i,\chi,q^{w_1},\zeta^{w_1}}(w_2 x)\\
&= \sum_{i=0}^{n}\binom{n}{i}[w_2]^i_q[w_1]^{n-i-1}_q\beta^{(h)}_{n-i,\chi,q^{w_1},\zeta^{w_1}}(w_2 x)\sum_{j=0}^{dw_1-1}\zeta^{w_2 j} q^{w_2(n-i+h)j}[j]^i_{q^{w_2}}\\
&= \sum_{i=0}^{n}\binom{n}{i}[w_2]^i_q[w_1]^{n-i-1}_q\beta^{(h)}_{n-i,\chi,q^{w_1},\zeta^{w_1}}(w_2 x)\, S^{(h)}_{n,i}(dw_1,\zeta^{w_2},q^{w_2}|\chi),
\end{aligned} \tag{2.6}$$

where

$$S^{(h)}_{n,i}(w_1,\zeta,q|\chi) = \sum_{j=0}^{w_1-1}\chi(j)\zeta^j q^{(n-i+h)j}[j]^i_q.$$

By the same method as (2.6), we get

$$\begin{aligned}
&[w_2]^{n-1}_q \sum_{j=0}^{dw_2-1}\chi(j)\zeta^{w_1 j} q^{hw_1 j}\int_X \chi(y)\zeta^{w_2 y} q^{w_2(h-1)y}\left[w_1 x + \frac{w_1}{w_2} j + y\right]^n_{q^{w_2}} d\mu_{q^{w_2}}(y)\\
&= \sum_{i=0}^{n}\binom{n}{i}[w_1]^i_q[w_2]^{n-i-1}_q\beta^{(h)}_{n-i,\chi,q^{w_2},\zeta^{w_2}}(w_1 x)\, S^{(h)}_{n,i}(dw_2,\zeta^{w_1},q^{w_1}|\chi).
\end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we have the following theorem.

Theorem 2.4. For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i-1}\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}^{(h)}(w_2x)\,S_{n,i}^{(h)}(dw_1,\zeta^{w_2},q^{w_2}|\chi)$$
$$=\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i-1}\beta_{n-i,\chi,q^{w_2},\zeta^{w_2}}^{(h)}(w_1x)\,S_{n,i}^{(h)}(dw_2,\zeta^{w_1},q^{w_1}|\chi).$$

Remark 2.5. Let $w_1, w_2 \in \mathbb{N}, n \geq 0$, and χ be the trivial character. Then we have

$$\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i-1}\beta_{n-i,q^{w_1},\zeta^{w_1}}^{(h)}(w_2x)\,S_{n,i}^{(h)}(w_1|\zeta^{w_2},q^{w_2})$$
$$=\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i-1}\beta_{n-i,q^{w_2},\zeta^{w_2}}^{(h)}(w_1x)\,S_{n,i}^{(h)}(w_2|\zeta^{w_1}q^{w_1}).$$

By Theorem 2.4, we obtain the interesting symmetric identity for Carlitz's generalized twisted (h, q)-Bernoulli numbers.

Corollary 2.6. For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$\sum_{i=0}^{n}\binom{n}{i}[w_2]_q^i[w_1]_q^{n-i-1}S_{n,i}^{(h)}(dw_1,\zeta^{w_2},q^{w_2}|\chi)\beta_{n-i,\chi,q^{w_1},\zeta^{w_1}}^{(h)}$$
$$=\sum_{i=0}^{n}\binom{n}{i}[w_1]_q^i[w_2]_q^{n-i-1}S_{n,i}^{(h)}(dw_2,\zeta^{w_1},q^{w_1}|\chi)\beta_{n-i,\chi,q^{w_2},\zeta^{w_2}}^{(h)}.$$

If we take $h = 1$ in all equations of this article, then [6] are the special case of our results.

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