Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 4 (2015), pp. 2627–2633

http://www.ripublication.com/gjpam.htm

Degenerate generalized q-Euler polynomials

Taekyun Kim
Department of Mathematics,
Kwangwoon University,
Seoul 139-701, Republic of Korea.
E-mail: tkkim@kw.ac.kr

Jin-Woo Park[1]
Department of Mathematics education,
Daegu University, Gyeongsan, 712-714, Korea.
E-mail: a0417001@knu.ac.kr

Jong-Jin Seo
Department of Applied Mathematics,
Pukyong National University,
Pusan, Republic of Korea.
E-mail: seo2011@pknu.ac.kr

Abstract

In this paper, we consider degenerate generalized q-Euler polynomials arising from p-adic fermionic q-integral on $\mathbb{Z}_p$ and give some identities of these polynomials.

AMS subject classification: 05A10, 05A19.
Keywords:

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \dfrac{1}{p}$.

[1]Corresponding author.

Let q be an indeterminate in $\mathbb{C}_p$ such that $|q-1|_p < p^{-\frac{1}{p-1}}$. The *$q$-analogue of number x* is defined as $[x]_q = \dfrac{1-q^x}{1-q}$. Note that $\lim_{q\to 1}[x]_q = x$.

Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then the *p-adic fermionic q-integral on $\mathbb{Z}_p$* is defined by Kim to be

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_{-q}\left(x+p^N\mathbb{Z}_p\right)\\ &= \lim_{N\to\infty}\frac{1+q}{1+q^{p^N}}\sum_{x=0}^{p^N-1} f(x)(-q)^x \\ &= \frac{[2]_q}{2}\lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)(-1)^x q^x, \text{ (see [4]).}\end{aligned} \tag{1.1}$$

Thus, by (1.1), we get

$$q\int_{\mathbb{Z}_p} f(x+1)d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = [2]_q f(0), \tag{1.2}$$

and

$$q^n\int_{\mathbb{Z}_p} f(x+n)d\mu_{-q}(x) + (-1)^{n-1}\int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x) = [2]_q\sum_{l=0}^{n-1} f(l)q^l(-1)^{n-1-l}, \tag{1.3}$$

where $n \in \mathbb{N}$ (see [2-7]).

It is known that the *q-Euler polynomials* are given by the generating function to be

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-q}(y) = \frac{[2]_q}{qe^t+1}e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{1.4}$$

When $x=0$, $E_{n,q} = E_{n,q}(0)$ are called *q-Euler numbers* (see [8]).

Recently, degenerate q-Euler polynomials are introduced by he generating function to be

$$\frac{[2]_q}{q(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\mathcal{E}_{n,q,\lambda}(x)\frac{t^n}{n!}, \text{ (see [4]).} \tag{1.5}$$

Note that $\lim_{\lambda\to 0}\mathcal{E}_{n,q,\lambda}(x) = E_{n,q}(x)$, $(n \geq 0)$, (see [1]).

For $d \in \mathbb{N}$ with $d \equiv$ (mod 2) and $(d, p) = 1$, we set

$$X = \lim_{\overleftarrow{N}}\mathbb{Z}/dp^N\mathbb{Z},\ X^* = \bigcup_{0<a<dp,\ a\nmid p}\left(a+dp\mathbb{Z}_p\right),$$

and

$$a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a \ (\bmod dp^N)\right\},$$

where $a \in \mathbb{Z}$ with $0 \leq a < dp^N - 1$.

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, let us assume that χ is a Dirichlet character with conductor d. Now, we consider the generalized q-Euler polynomials attached to χ which are given by the generating function to be

$$\begin{aligned}\int_X \chi(y)e^{(x+y)t}d\mu_{-q}(y) &= \left(\frac{[2]_q}{q^d e^{dt}+1}\sum_{a=0}^{d-1} q^q(-1)^a\chi(a)e^{at}\right)e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,q,\chi}(x)\frac{t^n}{n!}, \ \text{(see [4-6, 8]).}\end{aligned} \tag{1.6}$$

When $x = 0$, $E_{n,q,\chi} = E_{n,q,\chi}(0)$ are called *generalized q-Euler numbers attached to χ*.

In the viewpoint of (1.6), we consider degenerate generalized q-Euler polynomials which are derived from the fermionic q-integral on $\mathbb{Z}_p$. The purpose of this paper is to investigate some properties and identities of degenerate generalized q-Euler polynomials.

2. Degenerate generalized q-Euler polynomials

In this section, we assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, let χ be a Dirichlet's character with conductor d.

In the viewpoint of (1.6), we consider degenerate generalized q-Euler polynomials which are given by the generating function to be

$$\begin{aligned}&\int_X \chi(y)(1+\lambda t)^{\frac{x+y}{\lambda}}d\mu_{-q}(y) \\ &= \left(\frac{[2]_q}{q^d(1+\lambda t)^{\frac{d}{\lambda}}+1}\sum_{a=0}^{d-1} q^a(-1)^a\chi(a)(1+\lambda t)^{\frac{a}{\lambda}}\right)(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty}\mathcal{E}_{n,\chi,q,\lambda}(x)\frac{t^n}{n!},\end{aligned} \tag{2.1}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$.

From (1.5) and (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,\chi,q,\lambda}(x)\frac{t^n}{n!} &= \left(\frac{[2]_q}{q^d(1+\lambda t)^{\frac{d}{\lambda}}+1}\sum_{a=0}^{d-1}q^a(-1)^a\chi(a)(1+\lambda t)^{\frac{a+x}{\lambda}}\right)\\ &= \frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}q^a(-1)^a\chi(a)\left(\frac{[2]_{q^d}}{q^d(1+\lambda t)^{\frac{d}{\lambda}}+1}(1+\lambda t)^{\frac{d}{\lambda}\frac{a+x}{d}}\right)\\ &= \sum_{n=0}^{\infty}\left(\frac{[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}q^a(-1)^a\chi(a)\mathcal{E}_{n,q^d,\frac{\lambda}{d}}\left(\frac{a+x}{d}\right)\frac{d^nt^n}{n!}\right).\end{aligned} \tag{2.2}$$

Thus, by (2.2), we get

$$\mathcal{E}_{n,\chi,q,\lambda}(x) = \frac{d^n[2]_q}{[2]_{q^d}}\sum_{a=0}^{d-1}q^a(-1)^a\chi(a)\mathcal{E}_{n,q^d,\frac{\lambda}{d}}\left(\frac{a+x}{d}\right),\ (n \geq 0). \tag{2.3}$$

Therefore, by (2.3), we obtain the following theorem.

Theorem 2.1. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, $n \geq 0$, we have

$$\begin{aligned}\mathcal{E}_{n,\chi,q,\lambda}(x) &= \int_X \chi(y)(x+y|\lambda)_n d\mu_{-q}(y)\\ &= \frac{[2]_q}{[2]_{q^d}}d^n\sum_{a=0}^{d-1}\chi(a)q^a(-1)^a\mathcal{E}_{n,q^d,\frac{\lambda}{d}}\left(\frac{a+x}{d}\right),\end{aligned}$$

where

$$\begin{aligned}(x|\lambda)_n &= x(x-\lambda)\cdots(x-\lambda(n-1))\\ &= \lambda^n\left(\frac{x}{\lambda}\right)_n.\end{aligned}$$

For $n \geq 0$, we observe that

$$\begin{aligned}(x+y|\lambda)_n = \lambda^n\left(\frac{x+y}{\lambda}\right)_n &= \lambda^n\sum_{l=0}^{n}S_1(n,l)\left(\frac{x+y}{\lambda}\right)^l\\ &= \sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}(x+y)^l.\end{aligned} \tag{2.4}$$

By (2.4), we get

$$\begin{aligned}\int_X \chi(y)(x+y|\lambda)_n d\mu_{-q}(y) &= \sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}\int_X \chi(y)(x+y)^l d\mu_{-q}(y)\\ &= \sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}E_{l,q,\chi}(x),\ (n \geq 0),\end{aligned} \tag{2.5}$$

where $S_1(n,l)$ is the Stirling number of the first kind. Therefore, by (2.5), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$\mathcal{E}_{n,\chi,q,\lambda}(x) = \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l} E_{l,q,\chi}(x).$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.1), we get

$$\begin{aligned}\int_X \chi(y)e^{(x+y)t}d\mu_{-q}(y) &= \sum_{m=0}^{\infty}\mathcal{E}_{m,\chi,q,\lambda}(x)\frac{1}{m!}\left(\frac{1}{\lambda}(e^{\lambda t}-1)\right)^m \\ &= \sum_{m=0}^{\infty}\mathcal{E}_{m,\chi,q,\lambda}(x)\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n}{n!}t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\lambda^{n-m}S_2(n,m)\mathcal{E}_{m,\chi,q,\lambda}(x)\right)\frac{t^n}{n!},\end{aligned} \tag{2.6}$$

where $S_2(n,m)$ is the Stirling number of the second kind.

From (1.6), we note that

$$\begin{aligned}\int_X \chi(y)e^{(x+y)t}d\mu_{-q}(y) &= \left(\frac{[2]_q}{q^d e^{dt}+1}\right)\sum_{a=0}^{d-1}q^a(-1)^a\chi(a)e^{(a+x)t} \\ &= \sum_{n=0}^{\infty}E_{n,q,\chi}(x)\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, we have

$$E_{n,q,\chi}(x) = \sum_{m=0}^{n}\lambda^{n-m}S_2(n,m)\mathcal{E}_{m,\chi,q,\lambda}(x).$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$. From (1.3), we have

$$\begin{aligned}&q^d\int_X (x+d|\lambda)_n\chi(x)d\mu_{-q}(x) + \int_X (x|\lambda)_n\chi(x)d\mu_{-q}(x) \\ =&[2]_q\sum_{a=0}^{d-1}\chi(a)q^q(-1)^a(a|\lambda)_n,\ \ (n \geq 0).\end{aligned} \tag{2.8}$$

Therefore, by Theorem 2.1 and (2.8), we obtain the following theorem.

Theorem 2.4. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$, $n \geq 0$, we have

$$q^d \mathcal{E}_{n,\chi,q,\lambda}(d) + \mathcal{E}_{n,\chi,q,\lambda} = [2]_q \sum_{a=0}^{d-1} \chi(a) q^q (-1)^a (a|\lambda)_n,$$

where $\mathcal{E}_{n,\chi,q,\lambda} = \mathcal{E}_{n,\chi,q,\lambda}(0)$ are called *degenerate generalized q-Euler numbers attached to χ.*

Now, we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_{n,\chi,q,\lambda}(x) \frac{t^n}{n!} &= \left(\frac{[2]_q \sum_{a=0}^{d-1} q^a (-1)^a \chi(a)(1+\lambda t)^{\frac{a}{\lambda}}}{q^d (1+\lambda t)^{\frac{d}{\lambda}} + 1} \right) (1+\lambda t)^{\frac{x}{\lambda}} \\ &= \left(\sum_{m=0}^{\infty} \mathcal{E}_{n,\chi,q,\lambda} \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} (x|\lambda)_l \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} \mathcal{E}_{m,\chi,q,\lambda} (x|\lambda)_{n-m} \right) \frac{t^n}{n!}. \end{aligned} \tag{2.9}$$

Thus, by comparing the coefficients on the both sides, we obtain the following theorem.

Theorem 2.5. For $n \geq 0$, we have

$$\mathcal{E}_{n,\chi,q,\lambda}(x) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{E}_{m,\chi,q,\lambda} (x|\lambda)_{n-m}.$$

Now, we observe that

$$\begin{aligned} \frac{(-1)^n}{n!} \mathcal{E}_{n,\chi,q,\lambda} &= \frac{(-1)^n}{n!} \int_X \chi(x)(x|\lambda)_n d\mu_{-q}(x) \\ &= \lambda^n \int_X \binom{-\frac{x}{\lambda} + n - 1}{n} \chi(x) d\mu_{-q}(x) \\ &= \lambda^n \sum_{l=0}^{n} \binom{n-1}{l-1} \frac{(-1)^l}{\lambda^l l!} \int_X \chi(x)(x|-\lambda)_l d\mu_{-q}(x) \\ &= \sum_{l=0}^{n} \binom{n-1}{l-1} \lambda^{n-l} (-1)^l \frac{1}{l!} \mathcal{E}_{l,\chi,q,-\lambda} \\ &= \sum_{l=1}^{n} \binom{n-1}{l-1} \lambda^{n-l} (-1)^l \frac{\mathcal{E}_{l,\chi,q,-\lambda}}{l!}. \end{aligned} \tag{2.10}$$

Therefore, by (2.10), we obtain the following theorem.

Theorem 2.6. For $n \geq 0$, we have

$$\frac{(-1)^n}{n!}\mathcal{E}_{n,\chi,q,\lambda} = \sum_{l=1}^{n}\binom{n-1}{l-1}\lambda^{n-l}(-1)^l\frac{\mathcal{E}_{l,\chi,q,-\lambda}}{l!}.$$

Note that

$$\lim_{\lambda\to 0}\mathcal{E}_{n,\chi,q,\lambda}(x) = E_{n,q,\chi}(x),\ \ (n \geq 0).$$

References

[1] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Util. Math., **15** (1979), 51–88.

[2] D. V. Dolgy, D. S. Kim, T. Kim and J. J. Seo, *q-analogues of Boole polynomials*, Global J. Pure Appl. Math., **11** (2015), no. 4, 2323–2333.

[3] T. Kim, *Some identities on the q-Euler polynomials of higher-order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$*, Russ. J. Math. Phys., **16** (2009), 484–491.

[4] T. Kim, D. S. Kim and D. V. Dolgy *Degenerate q-Euler polynomials*, Adv. Difference Equ., **2015** (2015), 13662.

[5] D. S. Kim, T. Kim, D. V. Dolgy and T. Komatsu, *Barnes-type degenerate Bernoulli polynomials*, Adv. Stud. Contemp. Math., **25** (2015), 121–146.

[6] Q. M. Luo and F. Qi, *Relationship between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials*, Adv. Stud. Contemp. Math., **7** (2003), 11–18.

[7] S. H. Rim and J. Jeong, *On the modified q-Euler numbers of higher-order with weight*, Adv. Stud. Contemp. Math., **22** (2012), 93–98.

[8] E. Sen, *Theorems on Apostol-Euler polynomials of higher-order arising from Euler basis*, Adv. Stud. Contemp. Math., **23** (2013), 337–345.