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A Note on Interval Valued Fuzzy β-Subalgebras

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Abstract

In this paper, we intordude the notion of interval valued fuzzy β-subalgebras and investigate some of their properties.

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1. Introduction

In 1996, Y. Imai and Iseki ([3], [5], [4]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK algebras is a proper subclass of the class of BCI algebras. J. Neggers and Kim [6] introduced the notation of β-algebra where two operations are coupled in such a way to reflect the natural coupling, which exists between the usual group operation and its associated B-algebra. After L.A.

Zadeh proposed fuzzy sets [9], in 1994 R. Biswas [2] introduced Rosenfeld's fuzzy subgroups with interval valued membership functions. The fuzzy BCI-Subalgebras with Interval valued Membership functions was introduced by Houng, Jun and Kim [7]. Later Jun introduced Interval valued fuzzy sub algebras/ideals in BCK-algebra [8]. Recently Anasri et.al introduced fuzzy β-sub algebras of β-algebras [1]. With all these ideas in this paper we intend to introduce the notion of interval valued fuzzy β-subalgebra of a β-algebra.

2. Preliminaries

In this section we recall some basic definitions and results that are needed in the sequel.

Definition 2.1. A fuzzy set in X is defined as a function $\mu : X \to [0, 1]$. For each element x in X, $\mu(x)$ is called the membership value of $x \in X$ and X is a universal set.

Definition 2.2. [7] An interval valued fuzzy set (briefly i-v fuzzy set) A defined on X is given by

$$A = \{(x, [\mu_A^L(x), \mu_A^U(x)])\} \;\; \forall\, x \in X$$

(briefly denoted by $A = [\mu_A^L, \mu_A^U]$), where μ_A^L and μ_A^U are two fuzzy sets in X such that

$$\mu_A^L(x) \leq \mu_A^U(x) \;\; \forall\, x \in X.$$

Let

$$\overline{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)] \;\; \forall\, x \in X$$

and let $D[0, 1]$ denotes the family of all closed subintervals of $[0, 1]$. If $\mu_A^L(x) = \mu_A^U(x) = c$, say, where $0 \leq c \leq 1$, then we have $\overline{\mu}_A(x) = [c, c]$ which we also assume, for the sake of convenience, to belong to $D[0, 1]$. Thus $\overline{\mu}_A(x) \in D[0, 1] \;\; \forall\, x \in X$, and therefore the i-v fuzzy set A is given by

$$A = \{(x, \overline{\mu}_A(x))\} \;\; \forall\, x \in X,$$

where $\overline{\mu}_A : X \to D[0, 1]$.

Now let us define what is known as $refined\ mimimum$(briefly $rmim$) of two elements in $D[0, 1]$. We also define the symbols " $\geq$ ", " $\leq$ ", and " $=$ " in case of two elements in $D[0, 1]$.
Consider two elements $D_1 := [a_1, b_1]$ and $D_2 := [a_2, b_2] \in D[0, 1]$. Then we have

$$rmin(D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}];$$

$$D_1 \geq D_2$$

if and only if $a_1 \geq a_2,\ b_1 \geq b_2$; Similarly we may have $D_1 \leq D_2$ and $D_1 = D_2$.

Definition 2.3. A $\beta-$ algebra is a non-empty set X with a constant 0 and two binary operations $+$ and $-$ satisfying the following axioms:

(i) $x - 0 = x$

(ii) $(0 - x) + x = 0$

(iii) $(x - y) - z = x - (z + y) \ \forall x, y, z \in X$.

Example 2.4. Let $X = \{0, a, b, c\}$ be a set with constant 0 and binary operations $+$ and $-$ are defined on X by the following cayley's table

+	0	a	b	c
0	0	a	b	c
a	a	b	c	0
b	b	c	0	a
c	c	0	a	b

-	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then $(X, +, -, 0)$ is a β-algebra.

Definition 2.5. A non empty subset A of a β-algebra $(X, +, -, 0)$ is called a β-sub algebra of X, if

(i) $x + y \in A \ \forall \ x, y \in X$

(ii) $x - y \in A \ \forall \ x, y \in X$.

Example 2.6. In the above example of the β-algebra, the subset $A = \{0, b\}$ is a β-sub algebra of X.

Definition 2.7. Let μ be a fuzzy set in a β-algebra X. Then μ is called a fuzzy β-sub algebra of X if

(i) $\mu(x + y) \geq min\{\mu(x), \mu(y)\} \ \ \forall x, y \in X$

(ii) $\mu(x - y) \geq min\{\mu(x), \mu(y)\} \ \ \forall x, y \in X$.

3. Interval valued fuzzy β-subalgebra

In this section we introduce the notion of Interval valued fuzzy β-subalgebras of β-algebras and some elegant results are also discussed.

Definition 3.1. Let $\overline{\mu}_A$ be an i-v fuzzy subset in X. Then $\overline{\mu}_A$ is said to be interval valued fuzzy(i-v-fuzzy) β-sub algebra of X if

(i) $\overline{\mu}_A(x+y) \geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}\ \ \forall\, x, y \in X.$

(ii) $\overline{\mu}_A(x-y) \geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}\ \ \forall\, x, y \in X.$

Example 3.2. For the β-algebra defined in the example 2.4, the following i-v fuzzy set defined on X is a i-v fuzzy β-subalgebra of X.

$$\overline{\mu}_A = \begin{cases} [0.3, 0.7] : & x = 0 \\ [0.2, 0.6] : & x = a, c \\ [0.1, 0.5] : & x = b \end{cases}$$

Definition 3.3. Let $(X, +, -, 0)$ and $(Y, +, -, 0)$ be two β-algebras. A mapping $f : X \to Y$ is said to be a β-homomorphism, if

(i) $f(x+y) = f(x) + f(y)$

(ii) $f(x-y) = f(x) - f(y)\ \ \forall\, x, y \in X.$

Definition 3.4. If $\overline{\mu}_1$ and $\overline{\mu}_2$ are two i-v fuzzy $\beta-$subalgebras of$X$, then intersection $\overline{\mu_1 \cap \mu_2}$ of $\overline{\mu}_1$ and $\overline{\mu}_2$ is defined as

$$(\overline{\mu_1 \cap \mu_2})(x) \geq rmin\{\overline{\mu}_1(x), \overline{\mu}_2(x)\}$$

Theorem 3.5. An i-v fuzzy set $A = [\mu_A^L, \mu_A^U]$ in X is an i-v fuzzy β-subalgebra of X if and only if μ_A^L and μ_A^U are fuzzy β-subalgebras of X.

Proof. Let $A = [\mu_A^L, \mu_A^U]$ be an i-v fuzzy set in X. Suppose that μ_A^L and μ_A^U are fuzzy β-subalgebras of X. Let $x, y \in X$. Then

$$\begin{aligned} \overline{\mu}_A(x+y) &= [\mu_A^L(x+y), \mu_A^U(x+y)] \\ &\geq min\{[\mu_A^L(x), \mu_A^L(y)]\}, min\{[\mu_A^U(x), \mu_A^U(y)]\} \\ &= rmin\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\} \\ &= rmin\{[\overline{\mu}_A(x), \overline{\mu}_A(y)]\} \end{aligned}$$

Similarly,

$$\overline{\mu}_A(x-y) \geq rmin\{[\overline{\mu}_A(x), \overline{\mu}_A(y)]\}.$$

Hence A is an i-v fuzzy β-subalgebra of X.

Conversly, Assume that A is an i-v fuzzy $\beta-$subalgebra of X. Then for any $x, y \in X$, we have

$$\begin{aligned} [\mu_A^L(x+y), \mu_A^U(x+y)] &= \overline{\mu}_A(x+y) \\ &\geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\} \\ &= rmin\{[\mu_A^L(x), \mu_A^U(x)], [\mu_A^L(y), \mu_A^U(y)]\} \\ &= [min\{\mu_A^L(x), \mu_A^L(y)\}, min\{\mu_A^U(x), \mu_A^U(y)\}] \end{aligned}$$

It follows that

$$\mu_A^L(x+y) \geq min\{\mu_A^L(x), \mu_A^L(y)\}$$

and

$$\mu_A^U(x+y) \geq min\{\mu_A^U(x), \mu_A^U(y)\}$$

Hence μ_A^L and μ_A^U are fuzz $\beta-$subalgebra of X. ■

Theorem 3.6. If $\overline{\mu}_1$ and $\overline{\mu}_2$ be two i-v fuzzy β-sub algebra of X then $\overline{(\mu_1 \cap \mu_2)}$ is also an i-v fuzzy β-subalgebra of X.

Proof. For $x, y \in X$

$$\begin{aligned}
\overline{(\mu_1 \cap \mu_2)}(x+y) &= [(\mu_1 \cap \mu_2)^L(x+y), (\mu_1 \cap \mu_2)^U(x+y)] \\
&\geq [min\{(\mu_1 \cap \mu_2)^L(x), (\mu_1 \cap \mu_2)^L(y)\}, \\
&\quad min\{(\mu_1 \cap \mu_2)^U(x), (\mu_1 \cap \mu_2)^U(y)\}] \\
\overline{(\mu_1 \cap \mu_2)}(x+y) &\geq rmin\{\overline{(\mu_1 \cap \mu_2)}(x), \overline{(\mu_1 \cap \mu_2)}(y)\}
\end{aligned}$$

Similarly,

$$\overline{(\mu_1 \cap \mu_2)}(x-y) \geq rmin\{\overline{(\mu_1 \cap \mu_2)}(x), \overline{(\mu_1 \cap \mu_2)}(y)\}$$

Hence $\overline{\mu_1 \cap \mu_2}$ is an i-v fuzzy β-sub algebra of X. ■

Lemma 3.7. If A is an i-v fuzzy β-subalgebra of X then the following holds

(1) $\overline{\mu}_A(0) \geq \overline{\mu}_A(x) \ \ \forall x \in X$

(2) $\overline{\mu}_A(x) \leq \overline{\mu}_A(x^*) \leq \overline{\mu}_A(0) \ \ \forall x \in X$ where $x^* = 0 - x$.

Proof. (1) For every $x \in X$, we have

$$\begin{aligned}
\overline{\mu}_A(x) &= [\mu_A^L(x), \mu_A^U(x)] \\
&\leq [\mu_A^L(0), \mu_A^U(0)] = \overline{\mu}_A(0)
\end{aligned}$$

Hence

$$\overline{\mu}_A(0) \geq \overline{\mu}_A(x)$$

(2) For every $x \in X$, we have

$$\begin{aligned}
\overline{\mu}_A(x) &= [\mu_A^L(x), \mu_A^U(x)] \\
&\leq [\mu_A^L(x^*), \mu_A^U(x^*)] \\
&= \overline{\mu}_A(x^*)
\end{aligned}$$

Hence

$$\overline{\mu}_A(x) \leq \overline{\mu}_A(x^*) \leq \overline{\mu}_A(0).$$

■

Theorem 3.8. Let A be an i-v fuzzy β-subalgebra of X. If there is a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \overline{\mu}_A(x_n) = [1, 1]$$

then $\overline{\mu}_A(0) = [1, 1]$.

Proof. Since

$$\overline{\mu}_A(0) \geq \overline{\mu}_A(x) \ \ \forall\, x \in X,$$

we have $\overline{\mu}_A(0) \geq \overline{\mu}_A(x_n)$ for every positive integer n. Note that,

$$[1, 1] \geq \overline{\mu}_A(0) \geq \lim_{n\to\infty} \overline{\mu}_A(x_n) = [1, 1]$$

Hence $\overline{\mu}_A(0) = [1, 1]$. ∎

Theorem 3.9. Let A is an i-v fuzzy β-subalgebra of X, then the set

$$X_{\overline{\mu}_A} = \{x \in X / \overline{\mu}_A(x) = \overline{\mu}_A(0)\}$$

is a β-subalgebra of X.

Proof. Let x,y $\in X_{\overline{\mu}_A}$. Then

$$\overline{\mu}_A(x) = \overline{\mu}_A(0),$$
$$\overline{\mu}_A(y) = \overline{\mu}_A(0)$$

Now

$$\begin{aligned}\overline{\mu}_A(x + y) &\geq rmin\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}\\ &= rmin\{\overline{\mu}_A(0), \overline{\mu}_A(0)\}\\ &= \overline{\mu}_A(0)\end{aligned}$$

Similarly,

$$\overline{\mu}_A(x - y) \geq \overline{\mu}_A(0)$$

Hence

$$x + y, x - y \in X_{\overline{\mu}_A}$$
$$\therefore \quad X_{\overline{\mu}_A}$$

is a $\beta-$subalgebra of X. ∎

Definition 3.10. Let f be a mapping from a set X into a set Y. Let A be an i-v fuzzy set in X. Then the image of A, denoted by $f[A]$, is the i-v fuzzy set in Y with the membership function defined by

$$\overline{\mu}_{f[A]}(y) = \begin{cases} rsup\, \overline{\mu}_A(z), & \text{if} f^{-1}(y) \neq \emptyset \ \ \forall\, y \in Y,\\ [0, 0], & \text{otherwise}\end{cases}$$

where $f^{-1}(y) = \{x/f(x) = y\}$.

Lemma 3.11. Let f be a mapping a set X into a set Y. Let $m = [m^L, m^U]$ and $n = [n^L, n^U]$ be an i-v fuzzy sets in X and Y, respectively. Then

(i) $f^{-1}(n) = [f^{-1}(n^L), f^{-1}(n^U)]$

(ii) $f(m) = [f(m^L), f(m^U)]$.

Theorem 3.12. [1] Let $f : X \to Y$ be a homomorphism of a β-algebra X into a β-algebra Y. If μ is a fuzzy β-subalgebra of X with *supremum* property, then the image of μ, $f(\mu)$ defined by $f(\mu)(y) = f_{sup}(\mu(x))$ where $x \in f^{-1}(y)\ \ \forall\, y \in Y$, is a fuzzy β-subalgebra of Y.

Theorem 3.13. Let f be a homomorphism from a β-algebra X into a β-algebra Y. If A is an i-v fuzzy β-subalgebra of X, then the image $f[A]$ of A is an i-v fuzzy β-sub algebra of Y.

Proof. Assume that A is an i-v fuzzy β-subalgebra of X. $A = [\mu_A^L, \mu_A^U]$ is an i-v fuzzy β-subalgebra of X if and only if μ_A^L and μ_A^U are fuzzy β-subalgebras of X. It follows from theorem 3.12 that the images $f[\mu_A^L]$ and $f[\mu_A^U]$ are fuzzy β-subalgebras of Y. Combining theorem 3.5 and Lemma 3.11, we conclude that $f[A] = [f[\mu_A^L], f[\mu_A^U]]$ is an i-v fuzzy β-subalgebra of Y. ■

Definition 3.14. Let f be a mapping from a set X into a set Y. Let B be an i-v fuzzy set in Y. Then the inverse image of B, denoted by $f^{-1}(B)$, is the i-v fuzzy set in X with the membership function given by

$$\overline{\mu}_{f^{-1}(B)}(x) = \overline{\mu}_B(f(x))\ \ \forall\, x \in X.$$

Theorem 3.15. [1] Let $f : X \to Y$ be a homomorphism of a β-algebra X into a β-algebra Y. If μ is a fuzzy β-subalgebra of Y, then the preimage of μ, $f^{-1}(\mu)$ defined by $f^{-1}(\mu)(x) = \mu(f(x))\ \ \forall\, x \in X$ is a fuzzy β-subalgebra of X.

Theorem 3.16. Let f be a homomorphism from a β-algebra X into a β-algebra Y. If B is an i-v fuzzy β-subalgebra of Y, then the inverse image $f^{-1}[B]$ of B is an i-v fuzzy β-sub algebra of X.

Proof. Let $B = [\mu_B^L, \mu_B^U]$ be an i-v fuzzy β-subalgebra of Y. It follows from theorem 3.5 that μ_B^L and μ_B^U are fuzzy β-subalgebras of Y.

It follows from theorem 3.16 $f^{-1}[\mu_B^L]$ and $f^{-1}[\mu_B^U]$ are fuzzy β-sub algebra of X. Hence by Lemma 3.11 and Theorem 3.5, it follows that $f^{-1}[B] = [f^{-1}[\mu_B^L], f^{-1}[\mu_B^U]]$ is an i-v fuzzy β-subalgebra of X. ■

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