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Symmetric identities for Carlitz's twisted q-Bernoulli numbers and polynomials associated with p-adic invariant integral on $\mathbb{Z}_p$

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Abstract

In this paper, we study the symmetry for Carlitz's twisted q-Bernoulli numbers $\beta_{n,q,\zeta}$ and polynomials $\beta_{n,q,\zeta}(x)$. We obtain some interesting identities of the power sums and Carlitz's twisted q-Bernoulli polynomials $\beta_{n,q,\zeta}(x)$ using the symmetric properties for the p-adic invariant integral on $\mathbb{Z}_p$.

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Keywords: Bernoulli numbers and polynomials, q-Bernoulli numbers and polynomials, twisted q-Bernoulli numbers and polynomials, twisted q-Bernoulli numbers and polynomials.

1. Introduction

Many mathematicians have studies different kinds of the Euler, Bernoulli, Genocchi numbers and polynomials (see [1-7]). These numbers and polynomials play important roles in many different areas of mathematics such as number theory, combinatorics, special function and analysis. Recently, Y. Hu studied several identities of symmetry for Carlitz's q-Bernoulli numbers and polynomials in complex field(see [1]). D. Kim *et al.* [3] derived some identities of symmetry for generalized Carlitz's q-Bernoulli numbers and polynomials by using the p-adic integrals on $\mathbb{Z}_p$ in p-adic field. The purpose of this paper is to obtain some interesting identities of the power sums and Carlitz's twisted q-Bernoulli polynomials $\beta_{n,q,\zeta}(x)$ using the symmetric properties for the p-adic q-invariant integral on $\mathbb{Z}_p$.

Let p be a fixed prime number. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [1-4])}.$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \le 1$ in the present p-adic case. Let

$$g \in UD(\mathbb{Z}_p) = \{g|g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\}.$$

For $g \in UD(\mathbb{Z}_p)$ the p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_1(g) = \int_{\mathbb{Z}_p} g(x)d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x), \text{ (cf. [2])}. \tag{1.1}$$

We assume that $h \in \mathbb{Z}$. Let $T_p = \cup_{N\ge 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\zeta|w^{p^N} = 1\}$ is the cyclic group of order p^N. For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \zeta^x$ (cf. [6]).

2. Symmetric identities for Carlitz's twisted q-Bernoulli numbers and polynomials

D. Kim *et al.* [3] investigated interesting properties of symmetry p-adic invariant q-integral on $\mathbb{Z}_p$ for generalized q-Bernoulli polynomials. By using same method of [3], expect for obvious modifications, we obtain some symmetric properties for twisted q-Bernoulli polynomials. For $q \in \mathbb{C}_p$ with $|q-1|_p < p^{-\frac{1}{p-1}}$, the Carlitz's twisted q-Bernoulli polynomials $\beta_{n,q,\zeta}(x)$ are defined by

$$\beta_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \phi_\zeta(y) q^y [x+y]_q^n d\mu_1(y).$$

When $x = 0$, $\beta_{n,q,\zeta} = \beta_{n,q,\zeta}(0)$ is called the n-th twisted q-Bernoulli numbers $\beta_{n,q,\zeta}$.

Let w_1 and w_2 be natural numbers. Then, by (1.1), we obtain

$$\begin{aligned}
&\frac{1}{w_1}\int_{\mathbb{Z}_p} \zeta^{w_1 y} q^{w_1 y} e^{[w_1w_2x+w_2j+w_1y]_q t} d\mu_1(y) \\
&= \lim_{N\to\infty} \frac{1}{w_1}\frac{1}{w_2p^N} \sum_{y=0}^{w_2p^N-1} \zeta^{w_1y} q^{w_1y} e^{[w_1w_2x+w_2j+w_1y]_q t} q^{w_1y} \\
&= \lim_{N\to\infty} \frac{1}{w_1w_2p^N} \sum_{i=0}^{w_2-1} q^{w_1i}\zeta^{w_1i} \sum_{y=0}^{p^N-1} \zeta^{w_1w_2y} q^{w_1w_2y} e^{[w_1w_2x+w_2j+w_1i+w_1w_2y]_q t}
\end{aligned} \tag{2.1}$$

From (2.1), we can derive the following equation (2.2):

$$
\begin{aligned}
&\frac{1}{w_1}\sum_{j=0}^{w_1-1}\zeta^{w_2 j}q^{w_2 j}\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1 y}e^{[w_1w_2x+w_2j+w_1y]_q t}d\mu_1(y)\\
&=\lim_{N\to\infty}\frac{1}{w_1w_2p^N}\sum_{j=0}^{w_1-1}\sum_{i=0}^{w_2-1}\sum_{y=0}^{p^N-1}\zeta^{w_2 j}\zeta^{w_1 i}q^{w_2 j}q^{w_1 i}\\
&\qquad\times e^{[w_1w_2x+w_2j+w_1i+w_1w_2y)]_q t}\zeta^{w_1w_2y}q^{w_1w_2y}
\end{aligned}
\tag{2.2}
$$

By the same method as (2.2), we obtain

$$
\begin{aligned}
&\frac{1}{w_2}\sum_{j=0}^{w_2-1}\zeta^{w_1 j}q^{w_1 j}\int_{\mathbb{Z}_p}\zeta^{w_2 y}q^{w_2 y}e^{[w_1w_2x+w_1j+w_2y]_q t}d\mu_1(y)\\
&=\lim_{N\to\infty}\frac{1}{w_1w_2p^N}\sum_{j=0}^{w_2-1}\sum_{i=0}^{w_1-1}\sum_{y=0}^{p^N-1}\zeta^{w_1 i}\zeta^{w_2 i}q^{w_1 j}q^{w_2 i}\\
&\qquad\times e^{[w_1w_2x+w_1j+w_2i+w_1w_2y]_q t}\zeta^{w_1w_2y}q^{w_1w_2y}
\end{aligned}
\tag{2.3}
$$

Therefore, by (2.2) and (2.3), we have the following theorem.

Theorem 2.1. For $w_1, w_2 \in \mathbb{N}$, we have

$$
\begin{aligned}
&\frac{1}{w_1}\sum_{j=0}^{w_1-1}\zeta^{w_2 j}q^{w_2 j}\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1 y}e^{[w_1w_2x+w_2j+w_1y]_q t}d\mu_1(y)\\
&=\frac{1}{w_2}\sum_{j=0}^{w_2-1}\zeta^{w_1 j}q^{w_1 j}\int_{\mathbb{Z}_p}\zeta^{w_2 y}q^{w_2 y}e^{[w_1w_2x+w_1j+w_2y]_q t}d\mu_1(y).
\end{aligned}
\tag{2.4}
$$

By substituting Taylor series of e^{xt} into (2.4) and after elementary calculations, we obtain the following corollary.

Corollary 2.2. For $w_1, w_2 \in \mathbb{N}, n \geq 0$, we have

$$
\begin{aligned}
&\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{w_1-1}\zeta^{w_2 j}q^{w_2 j}\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1 y}\left[w_2x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_1(y)\\
&=\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{w_2-1}\zeta^{w_1 j}q^{w_1 j}\int_{\mathbb{Z}_p}\zeta^{w_2 y}q^{w_2 y}\left[w_1x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}^n d\mu_1(y).
\end{aligned}
$$

By Corollary 2.2, we have the following theorem.

Theorem 2.3. For $w_1, w_2 \in \mathbb{N}$, $n \geq 0$, we have

$$\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{w_1-1}\zeta^{w_2 j}q^{w_2 j}\beta_{n,q^{w_1},\zeta^{w_1}}\left(w_2 x+\frac{w_2}{w_1}j\right)$$
$$=\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{w_2-1}\zeta^{w_1 j}q^{w_1 j}\beta_{n,q^{w_2},\zeta^{w_2}}\left(w_1 x+\frac{w_1}{w_2}j\right).$$

By (2.5), we can derive the following equation(2.5):

$$\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1 y}\left[w_2 x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_1(y)$$
$$=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i[j]_{q^{w_2}}^i q^{w_2(n-i)j}\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1 y}\,[w_2 x+y]_{q^{w_1}}^{n-i}\,d\mu_1(y) \tag{2.5}$$
$$=\sum_{i=0}^{n}\binom{n}{i}\left(\frac{[w_2]_q}{[w_1]_q}\right)^i[j]_{q^{w_2}}^i q^{w_2(n-i)j}\beta_{n-i,\,q^{w_1},\zeta^{w_1}}\left(w_2 x\right).$$

By (2.5), and Theorem 2.3, we have

$$\frac{[w_1]_q^n}{w_1}\sum_{j=0}^{w_1-1}\zeta^{w_2 j}q^{w_2 j}\int_{\mathbb{Z}_p}\zeta^{w_1 y}q^{w_1 y}\left[w_2 x+\frac{w_2}{w_1}j+y\right]_{q^{w_1}}^n d\mu_1(y)$$
$$=\sum_{j=0}^{w_1-1}\zeta^{w_2 j}q^{w_2 j}\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}[j]_{q^{w_2}}^i q^{w_2(n-i)j}\beta_{n-i,q^{w_1},\zeta^{w_1}}\left(w_2 x\right)$$
$$=\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}\beta_{n-i,q^{w_1},\zeta^{w_1}}\left(w_2 x\right)\sum_{j=0}^{w_1-1}\zeta^{w_2 j}q^{w_2(n-i+1)j}[j]_{q^{w_2}}^i \tag{2.6}$$
$$=\sum_{i=0}^{n}\binom{n}{i}\frac{[w_2]_q^i[w_1]_q^{n-i}}{w_1}\beta_{n-i,q^{w_1},\zeta^{w_1}}\left(w_2 x\right)S_{n,i}(w_1,\zeta^{w_2},q^{w_2}),$$

where

$$S_{n,i}(w_1,\zeta,q)=\sum_{j=0}^{w_1-1}\zeta^j q^{(n-i+1)j}[j]_q^i.$$

By the same method as (2.6), we get

$$\frac{[w_2]_q^n}{w_2}\sum_{j=0}^{w_2-1}\zeta^{w_1 j}q^{w_1 j}\int_{\mathbb{Z}_p}\zeta^{w_2 y}q^{w_2 y}\left[w_1 x+\frac{w_1}{w_2}j+y\right]_{q^{w_2}}^n d\mu_1(y)$$
$$=\sum_{i=0}^{n}\binom{n}{i}\frac{[w_1]_q^i[w_2]_q^{n-i}}{w_2}\beta_{n-i,q^{w_2},\zeta^{w_2}}\left(w_1 x\right)S_{n,i}(w_2,\zeta^{w_1},q^{w_1}). \tag{2.7}$$

By (2.6) and (2.7), we have the following theorem.

Theorem 2.4. For $w_1, w_2 \in \mathbb{N}$, $n \geq 0$, we have

$$\sum_{i=0}^{n} \binom{n}{i} \frac{[w_2]_q^i [w_1]_q^{n-i}}{w_1} \beta_{n-i,q^{w_1},\zeta^{w_1}}(w_2 x)\, S_{n,i}(w_1, \zeta^{w_2}, q^{w_2})$$
$$= \sum_{i=0}^{n} \binom{n}{i} \frac{[w_1]_q^i [w_2]_q^{n-i}}{w_2} \beta_{n-i,q^{w_2},\zeta^{w_2}}(w_1 x)\, S_{n,i}(w_2, \zeta^{w_1}, q^{w_1}).$$

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