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On the partially degenerate Bernoulli polynomials of the first kind

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Abstract

In [2], L. Carlitz introduced the degenerate Bernoulli polynomials. In this paper, we consider the partially degenerate Bernoulli polynomials of the first kind which are different the Carlitz's degenerate Bernoulli polynomials.

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1. Introduction

The Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \text{ (see [1-17])}.$$

When $x = 0$, $B_n(0) = B_n$ are called Bernoulli numbers.

Let p be a fixed integer. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normally defined as $|p|_p = p^{-1}$.

Let $f(x)$ be continuous function on $\mathbb{Z}_p$. Then the p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu_0(x) &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(x)\mu_0(x+p^N\mathbb{Z}_p)\\ &= \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x), \text{ (see [9]).}\end{aligned} \tag{1}$$

From (1), we can derive the following equation:

$$\int_{\mathbb{Z}_p} f_n(x)d\mu_0(x) - \int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \sum_{l=0}^{n-1} f'(l), \quad (n\in\mathbb{N}), \tag{2}$$

where $f_n(x) = f(x+n)$, (see [9, 11]).

By (2), we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_0(y) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{3}$$

In this paper, we consider the partially degenerate Bernoulli polynomials of the first kind which are different the Carlitz's degenerate Bernoulli polynomials.

2. On the partially degenerate Bernoulli polynomials of the first kind

Let us assume that $\lambda, t \in \mathbb{C}_p$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$. Then, we consider the partially degenerate Bernoulli polynomials of the first kind which are given by the generating function to be

$$\frac{\log(1+\lambda t)^{1/\lambda}}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!}. \tag{4}$$

When $x = 0$, $B_{n,\lambda}(0) = B_{n,\lambda}$ are called the partially degenerate Bernoulli numbers of the first kind.

From (4), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0} B_{n,\lambda}(x)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{\log(1+\lambda t)^{1/\lambda}}{e^t-1}e^{xt}\\ &= \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}.\end{aligned}$$

Thus, we get

$$\lim_{\lambda\to 0} B_{n,\lambda}(x) = B_n(x), \ (n \geq 0).$$

From (4), we have

$$\begin{aligned}
\frac{\log(1+\lambda t)^{1/\lambda}}{e^t-1}e^{xt} &= \left(\frac{\log(1+\lambda t)}{\lambda t}\right)\left(\frac{t}{e^t-1}\right)e^{xt} \\
&= \left(\sum_{l=0}^{\infty}\frac{(-1)^l}{l+1}\lambda^l t^l\right)\left(\sum_{m=0}^{\infty}B_m(x)\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{n!(-1)^l\lambda^l}{(l+1)(n-l)!}B_{n-l}(x)\right)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{(-1)^l l!}{(l+1)}\lambda^l B_{n-l}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{5}$$

Therefore, by (4) and (5), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$B_{n,\lambda}(x) = \sum_{l=0}^{n}\binom{n}{l}\frac{l!}{(l+1)}(-\lambda)^l B_{n-l}(x).$$

By (4), we get

$$\begin{aligned}
\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!} &= \left(\frac{\log(1+\lambda t)^{1/\lambda}}{e^t-1}\right)e^{xt} \\
&= \left(\sum_{l=0}^{\infty}B_{n,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}x^m\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}x^{n-l}\right)\frac{t^n}{n!}.
\end{aligned} \tag{6}$$

From (6), we have

$$B_{n,\lambda}(x) = \sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}x^{n-l}, \ (n \geq 0). \tag{7}$$

It is not difficult to show that

$$B_{0,\lambda} = 1, \ (B_\lambda + 1)^n - B_{n,\lambda} = (n-1)!(-1)^{n-1}\lambda^{n-1}, \ \text{if } n \geq 1, \tag{8}$$

with the usual convention about replacing B_λ^n by $B_{n,\lambda}$.

Therefore, by (7) and (8), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$B_{n,\lambda}(x) = (B_\lambda + x)^n = \sum_{l=0}^{n} \binom{n}{l} B_{l,\lambda} x^{n-l}.$$

In particular,

$$B_{0,\lambda} = 1, (B_\lambda + 1)^n - B_{n,\lambda} = (n-1)!(-1)^{n-1}\lambda^{n-1}, (n \geq 1),$$

with the usual convention about replacing B_λ^n by $B_{n,\lambda}$.

Form (2), (3) and (4), we can derive the following equation:

$$\begin{aligned}
\frac{\log(1+\lambda t)^{1/\lambda}}{e^t - 1} e^{xt} &= \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (1+\lambda t)^{x_1} e^{(x_2+x)t} d\mu_0(x_1) d\mu_0(x_2) \\
&= \left(\sum_{l=0}^{\infty} \lambda^l \int_{\mathbb{Z}_p} (x_1)_l d\mu_0(x_1) \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} \int_{\mathbb{Z}_p} (x_2 + x)^m d\mu_0(x_2) \frac{t^m}{m!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \lambda^l \binom{n}{l} \int_{\mathbb{Z}_p} (x_1)_l d\mu_0(x_1) \int_{\mathbb{Z}_p} (x_2 + x)^{n-l} d\mu_0(x_2) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \lambda^l \binom{n}{l} B_{n-l}(x) \int_{\mathbb{Z}_p} (x_1)_l d\mu_0(x_1) \right) \frac{t^n}{n!},
\end{aligned} \tag{9}$$

where $(x_1)_l = x_1(x_1 - 1) \cdots (x_1 - l + 1)$.

It is known that Daehee polynomials are given by the generating function to be

$$\int_{\mathbb{Z}_p} (1+t)^{x_1+x} d\mu_0(x_1) = \frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}. \tag{10}$$

When $x = 0$, $D_n(0) = D_n$ are called Daehee numbers.

From (9) and (10), we have

$$\int_{\mathbb{Z}_p} (1+t)^{x_1} d\mu_0(x_1) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x_1)_n d\mu_0(x_1) \frac{t^n}{n!} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}, \tag{11}$$

where $(x_1)_n = x_1(x_1 - 1) \cdots (x_1 - n + 1)$.

Thus, by (11), we get

$$\int_{\mathbb{Z}_p} (x_1)_n d\mu_0(x_1) = D_n, (n \geq 0). \tag{12}$$

Therefore, by (9) and (12), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, we have

$$B_{n,\lambda}(x) = \sum_{l=0}^{n} \lambda^l \binom{n}{l} B_{n-l}(x) D_l.$$

Now, we observe that

$$\begin{aligned} \frac{\log(1+\lambda t)^{1/\lambda}}{e^t - 1} e^{xt} &= \frac{\log(1+\lambda t)^{1/\lambda}}{e^{dt}-1} \sum_{l=0}^{d-1} e^{(l+x)t} \\ &= \frac{1}{d} \sum_{l=0}^{d-1} \frac{\log(1+\frac{\lambda}{d} dt)^{d/\lambda}}{e^{dt}-1} e^{\left(\frac{l+x}{d}\right)dt} \\ &= \sum_{n=0}^{\infty} \left(d^{n-1} \sum_{l=0}^{d-1} B_{n,\lambda/d}\left(\frac{l+x}{d}\right)\right) \frac{t^n}{n!}, \ (d \in \mathbb{N}). \end{aligned} \tag{13}$$

Therefore, by (4) and (13), we obtain the following theorem.

Theorem 2.4. For $n \geq 0$ and $d \in \mathbb{N}$, we have

$$B_{n,\lambda}(x) = d^{n-1} \sum_{l=0}^{d-1} B_{n,\lambda/d}\left(\frac{l+x}{d}\right).$$

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