Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 4 (2015), pp. 2401-2406

http://www.ripublication.com

Quantum Algorithm for Maximum Integer *M*-dimensional Knapsack Problem by Numbering Method

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Abstract

A quantum algorithm for the maximum integer *m*-dimensional knapsack problem by a numbering method and its example are reported. When an optimal combination of *n* pieces of different weight luggage packed into each knapsack that a weight T_i [$1 \leq i \leq m$. *i* and *m* are integers.] can be put is requested, a computational complexity of a classical computation is $(2^n - 1)^m$. The computational complexity becomes about 4*mn* by the quantum algorithm that uses quantum phase inversion gates, quantum inversion about mean gates and the numbering method. Therefore, a polynomial time process becomes possible.

AMS Subject Classification: Primary 81-08; Secondary 68R05, 68W40.

Keywords: Quantum algorithm, maximum integer *m*-dimensional knapsack problem, numbering method, computational complexity, polynomial time.

1. Introduction

The methods for the very first steps towards building a quantum computer were developed by Haroche and Wineland [1]. On the other hand, Deutsch and Jozsa discovered the quantum algorithm of a high-speed process by a parallel computation that uses quantum entangled states [2-4]. After that, Shor found the method of solving the factoring in a polynomial time [3-5], and Grover showed the algorithm for the database search in a square root time [3, 6, 7]. A quantum algorithm for the 3-SAT problem by a numbering method has recently been reported by Fujimura [8]. Its computational complexity becomes a polynomial time. The maximum integer m-dimensional knapsack problem [9, 10] is examined by the numbering method this time. Therefore, its result is reported.

2. Maximum Integer *M*-dimensional Knapsack Problem

As for n pieces of different weight luggage, the maximum integer m-dimensional knapsack problem requests an optimal combination of the luggage packed into each knapsack that a weight T_i [$1 \leq i \leq m$. i and m are integers.] is assumed to be an upper bound [9, 10].

3. Quantum Algorithm

It is assumed that there are n pieces of different weight x_j [$1 \leq j \leq n$. j is an integer.], and the upper bound weight T_i [$1 \leq i \leq m$. i and m are integers.] of each knapsack, and $a_{i,j}$ is 0 or 1. When a combination ($a_{1,1}$, $a_{1,2}$, $\cdots$, $a_{i,j}$, $\cdots$, $a_{m,n\text{-}1}$, $a_{m,n}$) is labeled (a_1, a_2, $\cdots$, $a_{(i\text{-}1)n+j}$, $\cdots$, $a_{(m\text{-}1)n+n\text{-}1\,=\,mn\text{-}1}$, $a_{(m\text{-}1)n+n\,=\,mn}$), $V(N)$ is the N-th (a_1, a_2, $\cdots$, $a_{(i\text{-}1)n+j}$, $\cdots$, $a_{mn\text{-}1}$, a_{mn}) [$0 \leq N \leq 2^{mn}$-1. N is an integer. $V(0)$ is (0, 0, $\cdots$, 0, $\cdots$, 0, 0). $V(2^{mn}$-1) is (1, 1, $\cdots$, 1, $\cdots$, 1, 1).]. This method is named the numbering method for this problem. g is the minimum integer that follows $2^{mn}/1 \leq 4^g = 2^{2g}$, because a number of combinations of an answer is at least 1.

First of all, quantum registers $|a_{i,j}>$, $|b_i>$, $|c>$, $|d>$, $|e_1>$ and $|e_2>$ are prepared. States of $|a_{i,j}>$, $|b_i>$, $|c>$, $|d>$, $|e_1>$ and $|e_2>$ are $a_{i,j}$, b_i, c, d, e_1 and e_2, respectively.

Step 1: Each quantum bit [=qubit] of $|a_{i,j}>$, $|b_i>$, $|c>$, $|d>$, $|e_1>$ and $|e_2>$ is set $|0>$.

Step 2: The Hadamard gate $\boxed{\text{H}}$ [3, 4] acts on each qubit of $|a_{i,j}>$. It changes them for entangled states. The total states are 2^{mn} [= W_0].

Step 3: It is assumed that a quantum gate (A_i) changes $|b_i>$ for $|b_i + a_{i,j}\, x_j>$ at $|a_{i,j}>$, and it changes $|c>$ for $|a_1, a_2, \cdots, a_{(i-1)n+j}>$ at $|a_{i,j} = a_{(i-1)n+j}>$. These actions are repeated sequentially from $|a_{1,1}>$ to $|a_{m,n}>$. Therefore, $|b_i>$ is $|\sum_{j=1\to n} a_{i,j}\, x_j>$, and $|c>$ is $|a_1, a_2, \cdots, a_{mn}>$.

Step 4: It is assumed that a quantum gate (B) changes $|d>$ for $|d + 1>$ at $b_i = T_i$, or it changes $|d>$ for $|d + 0>$ in the others of b_i. These actions are repeated sequentially from $|b_1>$ to $|b_m>$.

Step 5: It is assumed that a quantum gate (C) changes $|e_1>$ for $|V(2^{mn} - 1)>$ at $d = m$, or it changes $|e_1>$ for $|a_1, a_2, \cdots, a_{nm-1}, a_{nm}>$ in the others of d.

Step 6: It is assumed that a quantum gate (D_1) changes $|e_2>$ for $|1>$ in $V(0) \le e_1 \le V((2^{mn}/4) - 2)$ or $e_1 = V(2^{mn} - 1)$, or it changes $|e_2>$ for $|0>$ in the others of e_1. As a target state for $|e_2>$ is 1, quantum phase inversion gates (PI) and quantum inversion about mean gates (IM) [3, 6, 7] act on $|e_2>$. The number of the data that is included in $V(0) \le e_1 \le V((2^{mn}/4) - 2)$ or $e_1 = V(2^{mn} - 1)$ is $W_1 \approx 2^{mn}/4$. When R_1 is the minimum even integer that is $(W_0/W_1)^{1/2} \approx (2^{mn}/(2^{mn}/4))^{1/2}$ or more, the total number that (PI) and (IM) act on $|e_2>$ is $R_1 \approx 2$, because they are a couple. Next, an observation gate (OB) observes $|e_2>$, and the data of W_1 remain. Similarly, (D_h) [$2 \le h \le g - 1$. h is the integer.] changes $|e_2>$ for $|1>$ in $V(0) \le e_1 \le V((2^{mn}/4^h) - 2)$ or $e_1 = V(2^{mn} - 1)$, or it changes $|e_2>$ for $|0>$ in the others of e_1. As the target state for $|e_2>$ is 1, (PI) and (IM) act on $|e_2>$. The number of the data that is included in $V(0) \le e_1 \le V((2^{mn}/4^h) - 2)$ or $e_1 = V(2^{mn} - 1)$ is $W_h \approx 2^{mn}/4^h$. When R_h is the minimum even integer that is $(W_{h-1}/W_h)^{1/2} \approx ((2^{mn}/4^{h-1})/(2^{mn}/4^h))^{1/2}$ or more, the total number that (PI) and (IM) act on $|e_2>$ is $R_i \approx 2$. Next, (OB) observes $|e_2>$, and the data of W_h remain. These actions are repeated sequentially from 2 to $g - 1$ at h. (D_g) changes $|e_2>$ for $|1>$ at $e_1 = V(2^{mn} - 1)$, or it changes $|e_2>$ for $|0>$ in the others of e_1. As the target state for $|e_2>$ is 1, (PI) and (IM) act on $|e_2>$. The number of the data that is included at $e_1 = V(2^{mn} - 1)$ is $W_g \approx 2^{mn}/4^g \approx 1$. When R_g is the minimum even integer that is $(W_{g-1}/W_g)^{1/2} \approx ((2^{mn}/4^{g-1})/(2^{mn}/4^g))^{1/2}$ or more, the total number that (PI) and (IM) act on $|e_2>$ is $R_g \approx 2$. Next, (OB) observes $|a_{i,j}>$, $|b_i>$, $|c>$, $|d>$, $|e_1>$ and $|e_2>$, and one of the data of W_g remains. Therefore, one example of combinations that are $b_i = T_i$ is obtained.

4. Numerical Computation

It is assumed that there are $n = 5$, $m = 3$, $1 \le j \le 5$, $1 \le i \le 3$, $x_1 = 3$, $x_2 = 2$, $x_3 = 7$, $x_4 = 10$, $x_5 = 13$, $T_1 = 19$, $T_2 = 16$, $T_3 = 17$, $g = 8$, $V(2^{15} - 1 = 32767) = (1, 1, 1, 1, 1, 1, 1, 1,$

1, 1, 1, 1, 1, 1, 1), $V((2^{15}/4) - 2 = 8190) = (0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0)$, $V((2^{15}/4^2) - 2 = 2046) = (0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0)$, $V((2^{15}/4^3) - 2 = 510) = (0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0)$, $V((2^{15}/4^4) - 2 = 126) = (0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0)$, $V((2^{15}/4^5) - 2 = 30) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0)$, $V((2^{15}/4^6) - 2 = 6) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0)$ and $V((2^{15}/4^7) - 2 = 0) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

First of all, $|a_{i,j}>$, $|b_i>$, $|c>$, $|d>$, $|e_1>$ and $|e_2>$ are prepared. States of $|a_{i,j}>$, $|b_i>$, $|c>$, $|d>$, $|e_1>$ and $|e_2>$ are $a_{i,j}$, b_i, c, d, e_1 and e_2, respectively.
Step 1: Each qubit of $|a_{i,j}>$, $|b_i>$, $|c>$, $|d>$, $|e_1>$ and $|e_2>$ is set $|0>$.
Step 2: $\boxed{H}$ acts on each qubit of $|a_{i,j}>$. It changes them for entangled states. The total states are $2^{3\cdot 5} = 2^{15}$ [$= W_0$].
Step 3: (A_i) changes $|b_i>$ for $|b_i + a_{i,j}\, x_i>$ at $|a_{i,j}>$, and it changes $|c>$ for $|a_1, a_2, \cdots, a_{5(i-1)+j}>$ at $|a_{i,j} = a_{5(i-1)+j}>$. These actions are repeated sequentially from $|a_{1,1}>$ to $|a_{3,5}>$. Therefore, $|b_i>$ is $|a_{i,1}\, x_1 + a_{i,2}\, x_2 + \cdots + a_{i,5}\, x_5>$, and $|c>$ is $|a_1, a_2, \cdots, a_{15}>$.
Step 4: (B) changes $|d>$ for $|d + 1>$ at $b_i = T_i$, or it changes $|d>$ for $|d + 0>$ in the others of b_i. These actions are repeated sequentially from $|b_1>$ to $|b_3>$.
Step 5: (C) changes $|e_1>$ for $|e_1> = |V(2^{15} - 1 = 32767)> = |1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1>$ at $d = 3$, or it changes $|e_1>$ for $|e_1> = |a_1, a_2, \cdots, a_{15}>$ in the others of d.
Step 6: (D_1) changes $|e_2>$ for $|1>$ in $V(0) \le e_1 \le V((2^{15}/4) - 2 = 8190)$ or $e_1 = V(2^{15} - 1)$, or it changes $|e_2>$ for $|0>$ in the others of e_1. As a target state for $|e_2>$ is 1, (PI) and (IM) act on $|e_2>$. The number of the data that is included in $V(0) \le e_1 \le V(8190)$ or $e_1 = V(2^{15} - 1)$ is $W_1 \approx 2^{15}/4$. When R_1 is the minimum even integer that is $(W_0/W_1)^{1/2} \approx (2^{15}/(2^{15}/4))^{1/2} = 2 \le 2 = R_1$, the total number that (PI) and (IM) act on $|e_2>$ is $R_1 \approx 2$. Next, (OB) observes $|e_2>$, and the data of W_1 remain. Similarly, (D_h) [$2 \le h \le 7$. h is the integer.] changes $|e_2>$ for $|1>$ in $V(0) \le e_1 \le V((2^{15}/4^h) - 2)$ [$h = 2$; $V((2^{15}/4^2) - 2 = 2046)$, $h = 3$; $V((2^{15}/4^3) - 2 = 510)$, $h = 4$; $V((2^{15}/4^4) - 2 = 126)$, $h = 5$; $V((2^{15}/4^5) - 2 = 30)$, $h = 6$; $V((2^{15}/4^6) - 2 = 6)$, $h = 7$; $V((2^{15}/4^7) - 2 = 0)$] or $e_1 = V(2^{15} - 1)$, or it changes $|e_2>$ for $|0>$ in the others of e_1. As the target state for $|e_2>$ is 1, (PI) and (IM) act on $|e_2>$. The number of the data that is included in $V(0) \le e_1 \le V((2^{15}/4^h) - 2)$ or $e_1 = V(2^{15} - 1)$ is $W_h \approx 2^{15}/4^h$. When R_h is the minimum even integer that is $(W_{h-1}/W_h)^{1/2} \approx ((2^{15}/4^{h-1})/(2^{15}/4^h))^{1/2} = 2 \le 2 = R_h$, the total number that (PI) and (IM) act on $|e_2>$ is $R_h \approx 2$. Next, (OB) observes $|e_2>$, and the data of W_h remain. These actions are repeated sequentially from 2 to 7 at h. (D_8) changes $|e_2>$ for $|1>$ at $e_1 = V(2^{15} - 1)$, or it changes $|e_2>$ for $|0>$ in the others of e_1. As the target state for $|e_2>$ is 1, (PI) and (IM) act on

$|e_2>$. The number of the data that is included at $e_1 = V(2^{15} - 1)$ is $W_8 \approx 2^{15}/4^8 \approx 1$. When R_8 is the minimum even integer that is $(W_7/W_8)^{1/2} \approx ((2^{15}/4^7)/(2^{15}/4^8))^{1/2} = 2 \leq 2 = R_8$, the total number that (*PI*) and (*IM*) act on $|e_2>$ is $R_8 \approx 2$. Next, (*OB*) observes $|a_{i,j}>$, $|b_i>$, $|c>$, $|d>$, $|e_1>$ and $|e_2>$, and one of the data of W_8 remains. For example, when $a_{1,1}$, $a_{1,2}$, $a_{1,3}$, $a_{1,4}$, $a_{1,5}$, $a_{2,1}$, $a_{2,2}$, $a_{2,3}$, $a_{2,4}$, $a_{2,5}$, $a_{3,1}$, $a_{3,2}$, $a_{3,3}$, $a_{3,4}$, $a_{3,5}$, b_1, b_2, b_3, c, d, e_1 and e_2 are 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0, 19, 16, 17, (0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0), 3, $V(2^{15} - 1)$ and 1, respectively, it is obtained that combinations of the 1st, 2nd and 3rd knapsacks are (2, 7, 10), (3, 13) and (7, 10), respectively.

5. Discussion and Summary

The computational complexity of this quantum algorithm [= S] becomes the following. In the order of the actions by the gates, the number of them is mn at $\boxed{H}$, mn at (A_i), m at (B), n at (C), g at (D_h) [$1 \leq h \leq g$. h is the integer.], $\sum_{h=1 \to g} R_h \approx 2g$ at (*PI*) and (*IM*), and g at (*OB*). Therefore, S becomes $2mn + m + 2 + 4g$. In the example of the numerical computation, S is 67. The computational complexity of the classical computation [= Z] is $(2^n - 1)^m = (2^5 - 1)^3 = 29791$. After all, S/Z becomes about 1/445. When n is large enough, S becomes about $4mn$, where g is about $mn/2$. And then, S/Z is about $4mn/2^{mn} \approx mn/2^{mn}$. For example, as for $n = 100$ and $m = 3$, S/Z is about $300/2^{300} \approx 1/10^{88}$. Therefore, the polynomial time process becomes possible.

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