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On Contra-b-$\mathcal{I}$-Continuous Functions

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Abstract

In this paper, b-$\mathcal{I}$-closed sets and b-$\mathcal{I}$-open sets are used to define and investigate a new class of functions called contra- b-$\mathcal{I}$-continuous functions in ideal topological spaces. Relationships between this new class and other classes of functions are established.

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1. Introduction

The subject of ideals in topological spaces has been introduced and studied by Kuratowski [10] and Vaidyanathasamy [18]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, then the set operator $(.)^\star : \mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [18] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^\star(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$, for every open set U containing $x\}$. A Kuratowski closure operator $\mathrm{Cl}^\star(.)$ for a topology $\tau^\star(\tau, \mathcal{I})$ called the $\star$-topology, finer than τ is defined by $\mathrm{Cl}^\star(A) = A \cup A^\star(\tau, \mathcal{I})$. When there is no chance for confusion, $A^\star(\tau, \mathcal{I})$ is denoted by $A^\star$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ will denote the closure and interior of A in (X, τ), respectively. A subset A of (X, τ) is said to be b-open [2] or γ-open [8] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$. The complement of a b-open set is called a b-closed set. The aim of this paper is to give a new class of functions called contra-b-$\mathcal{I}$-continuous in an ideal topological space. Some characterizations and several basic properties of this class of functions are obtained. Also we define b-$\mathcal{I}$-normal spaces using b-$\mathcal{I}$-open sets and give characterizations and properties of such spaces.

2. Preliminaries

A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-open if $S \subset \mathrm{Cl}^\star(\mathrm{Int}(S)) \cup \mathrm{Int}(\mathrm{Cl}^\star(S))$. The complement of a b-$\mathcal{I}$-open set is called a b-$\mathcal{I}$-closed set. The intersection of all b-$\mathcal{I}$-closed (resp. b-closed) sets containing S is called the b-$\mathcal{I}$-closure (resp. b-closure) of S and is denoted by $b\mathcal{I}\,\mathrm{Cl}(S)$ (resp. $b\,\mathrm{Cl}(S)$). The family of all b-$\mathcal{I}$- open (resp. b-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ is denoted by $B\mathcal{I}O(X)$ (resp. $B\mathcal{I}C(X)$). The family of all b-$\mathcal{I}$-open (resp. b-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $B\mathcal{I}O(X, x)$ (resp. $B\mathcal{I}C(X, x)$). A topological space (X, τ) is said to be a b-T_2 space [16] if for each pair of distinct points $x, y \in X$, there exist $U, V \in bO(X)$ containing x and y, respectively, such that $U \cap V = \varnothing$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra-continuous [7] (resp. contra-b-continuous [5] or contra-γ-continuous [14], γ-continuous [8]) if $f^{-1}(V)$ is a closed (resp. b-closed or γ-closed, γ-open) set in X for every open set V of Y. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be b-$\mathcal{I}$-continuous [4] if $f^{-1}(V)$ is a b-$\mathcal{I}$-open set in $(X, \tau, \mathcal{I})$ for every open set V of Y.

Definition 2.1. Let A be a subset of a topological space (X, τ). The set $\cap\{U \in \tau | A \subset U\}$ is called the kernel of A [13] and is denoted by $\ker(A)$.

Lemma 2.2. [9] The following properties hold for subsets A, B of a space X:

1. $x \in \ker(A)$ if and only if $A \cap F \neq \emptyset$ for any closed set F of X containing x;
2. $A \subset \ker(A)$ and $A = \ker(A)$ if A is open in X;
3. If $A \subset B$, then $\ker(A) \subset \ker(B)$.

3. Contra-b-$\mathcal{I}$-Continuous functions

Definition 3.1. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is called contra-b-$\mathcal{I}$-continuous if $f^{-1}(V)$ is a b-$\mathcal{I}$-open set in $(X, \tau, \mathcal{I})$ for every closed set V of Y. Equivalently, f is contra-b-$\mathcal{I}$-continuous if and only if for each $x \in X$ and closed set F in Y containing $f(x)$, there exists a b-$\mathcal{I}$-open set U containing x such that $f(U) \subset F$.

Clearly, every contra-continuous function is contra-b-$\mathcal{I}$-continuous and every contra-b-$\mathcal{I}$-continuous function is contra-b-continuous but the converses are not true.

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\sigma = \{\emptyset, \{a\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ is contra-b-$\mathcal{I}$-continuous but not contra-continuous.

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{a, c\}, X\}$, $\sigma = \{\emptyset, \{b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ is contra-b-continuous but not contra-b-$\mathcal{I}$-continuous.

Remark 3.4. The following example show thats b-$\mathcal{I}$-continuity and contra-b-$\mathcal{I}$-continuity are independent concepts.

Example 3.5. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (X, \tau)$ is b-$\mathcal{I}$-continuous but not contra-b-$\mathcal{I}$-continuous. Also the function $f : (X, \tau, \mathcal{I}) \to (X, \tau)$ defined by $f(a) = b$, $f(b) = c$ and $f(c) = a$ is contra b-$\mathcal{I}$-continuous but not b-$\mathcal{I}$-continuous.

However, we have the following.

Theorem 3.6. If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra-b-$\mathcal{I}$- continuous and Y is a regular space, then f is b-$\mathcal{I}$-continuous.

Proof. Let x be an arbitrary point of X and V an open set of Y containing $f(x)$. Since Y is regular, there exists an open set W in Y containing $f(x)$ such that $\mathrm{Cl}(W) \subset V$. Since f is contra-b-$\mathcal{I}$-continuous, so by definition, there exists $U \in B\mathcal{I}O(X, x)$ such that $f(U) \subset \mathrm{Cl}(W)$. Then $f(U) \subset \mathrm{Cl}(W) \subset V$. Hence, by Theorem 4.6 of [1], f is b-$\mathcal{I}$-continuous. ■

Corollary 3.7. ([14], Theorem 3.2) If a function $f : (X, \tau) \to (Y, \sigma)$ is contra-b-continuous and Y is a regular space, then f is b-continuous.

Proof. The proof follows from Theorem 3.6, if $\mathcal{I} = \{\emptyset\}$. ■

Theorem 3.8.

(1) *A* function $f : (X, \tau, \{\emptyset\}) \to (Y, \sigma)$ is contra-b-$\mathcal{I}$-continuous if and only if it is contra-b-continuous.

(2) *A* function $f : (X, \tau, \mathcal{N}) \to (Y, \sigma)$ is contra-b-$\mathcal{I}$-continuous if and only if it is contra-b-continuous ($\mathcal{N}$ is the ideal of all nowhere dense sets).

(3) *A* function $f : (X, \tau, \mathcal{P}(X)) \to (Y, \sigma)$ is contra-b-$\mathcal{I}$-continuous if and only if it is contra-continuous.

Proof. It follows from Proposition 2 of [4]. ■

Definition 3.9. [6] Let A and B be subsets of an ideal topological space $(X, \tau, \mathcal{I})$ such that $A \subset B \subset X$. Then $(B, \tau_{|B}, \mathcal{I}_{|B})$ is an ideal topological space with an ideal $\mathcal{I}_{|B} = \{I \in \mathcal{I} | I \subset B\} = \{I \cap B | I \in \mathcal{I}\}$.

Theorem 3.10. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a contra-b-$\mathcal{I}$-continuous function and A an open subset of $(X, \tau, \mathcal{I})$. Then the restriction $f_{|A}$ is contra-b-$\mathcal{I}$-continuous.

Proof. It follows from Theorem 3.15 of [1]. ■

Corollary 3.11. ([5], Theorem 2.16) Let $f : (X, \tau) \to (Y, \sigma)$ be a contra-b-continuous function and A an α-open subset of (X, τ). Then the restriction $f_{|A}$ is contra-b-continuous.

Proof. The proof follows from Theorem 3.10, if $\mathcal{I} = \{\emptyset\}$. ■

Theorem 3.12. The following statements are equivalent for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$:

1. f is contra-b-$\mathcal{I}$-continuous;
2. $f(b\mathcal{I}\,\mathrm{Cl}(A))) \subset \ker(f(A))$ for every subset A of X;
3. $b\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\ker(B))$ for every subset B of Y.

Proof.

(1) ⇒ (2): Let A be any subset of X. Suppose that $y \notin \ker(f(A))$. Then by Lemma 2.2 there exists a closed set F of Y containing y such that $f(A) \cap F = \emptyset$. Thus, we have $A \cap f^{-1}(F) = \emptyset$ and $b\mathcal{I}\,\mathrm{Cl}(A) \cap f^{-1}(F) = \emptyset$. Therefore, we obtain $f(b\mathcal{I}\,\mathrm{Cl}(A)) \cap F = \emptyset$ and $y \notin f(b\mathcal{I}\,\mathrm{Cl}(A))$. This implies that $f(b\mathcal{I}\,\mathrm{Cl}(A)) \subset \ker(f(A))$.

(2) ⇒ (3): Let B be any subset of Y. By (ii) and Lemma 2.2, we have $f(b\mathcal{I}\,\mathrm{Cl}(f^{-1}(B))) \subset \ker(f(f^{-1}(B))) \subset \ker(B)$ and $b\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\ker(B))$.

(3) ⇒ (1): Let V be any open subset of Y. Then by Lemma 2.2, we have $b\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\ker(V)) = f^{-1}(V)$ and $b\mathcal{I}\,\mathrm{Cl}(f^{-1}(V)) = f^{-1}(V)$. This show that $f^{-1}(V)$ is b-$\mathcal{I}$-closed in $(X, \tau, \mathcal{I})$. ■

Corollary 3.13. ([14], Theorem 3.1) The following statements are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$:

1. f is contra-b-continuous;
2. $f(\gamma\,\mathrm{Cl}(A)) \subset \ker(f(A))$ for every subset A of X;
3. $\gamma\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\ker(B))$ for every subset B of Y.

Proof. The proof follows from Theorem 3.12, if $\mathcal{I} = \{\emptyset\}$. ■

Theorem 3.14. Let $(X, \tau, \mathcal{I})$ be any ideal topological space and let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function and $g : X \to X \times Y$ be the graph function, given by $g(x) = (x, f(x))$

for every $x \in X$. Then f is contra-b-$\mathcal{I}$-continuous if and only if g is contra-b-$\mathcal{I}$-continuous.

Proof. Let $x \in X$ and let W be a closed subset of $X \times Y$ containing $g(x)$. Then $W \cap (\{x\} \times Y)$ is closed in $\{x\} \times Y$ containing $g(x)$. Also $\{x\} \times Y$ is homeomorphic to Y. Hence $\{y \in Y | (x, y) \in W\}$ is a closed subset of Y. Since f is contra-b-$\mathcal{I}$-continuous, $\cup\{f^{-1}(y) | (x, y) \in W\}$ is a b-$\mathcal{I}$-open subset of $(X, \tau, \mathcal{I})$. Further, $x \in \cup\{f^{-1}(y) | (x, y) \in W\} \subset g^{-1}(W)$. Hence $g^{-1}(W)$ is b-$\mathcal{I}$-open. Then g is contra-b-$\mathcal{I}$-continuous. Conversely, let F be a closed subset of Y. Then $X \times F$ is a closed subset of $X \times Y$. Since g is contra-b-$\mathcal{I}$-continuous, $g^{-1}(X \times F)$ is a b-$\mathcal{I}$-open subset of X. Also, $g^{-1}(X \times F) = f^{-1}(F)$. Hence f is contra-b-$\mathcal{I}$-continuous. ■

Corollary 3.15. ([5], Theorem 2.13) Let (X, τ) be any topological space and let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : X \to X \times Y$ be the graph function, given by $g(x) = (x, f(x))$ for every $x \in X$. Then f is contra-b-continuous if and only if g is contra-b-continuous.

Proof. The proof follows from Theorem 3.14, if $\mathcal{I} = \{\emptyset\}$. ■

Definition 3.16. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly continuous [11] if for each $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset \mathrm{Cl}(V)$.

Theorem 3.17. If $f : (X, \tau) \to (Y, \sigma)$ is weakly continuous and $g : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra-b-$\mathcal{I}$-continuous and Y is Urysohn, then the set $E = \{x \in X : f(x) = g(x)\}$ is b-$\mathcal{I}$-closed.

Proof. If $x \in X \backslash E$, then it follows that $f(x) \neq g(x)$. Since Y is Urysohn, there exist open sets V and W such that $f(x) \in V$, $g(x) \in W$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since f is weakly continuous and g is contra-b-$\mathcal{I}$-continuous, there exists an open set U containing x and a b-$\mathcal{I}$-open set G containing x such that $f(U) \subset \mathrm{Cl}(V)$ and $g(G) \subset \mathrm{Cl}(W)$. Set $H = U \cap G$. By Theorem 3.15 of [1], H is a b-$\mathcal{I}$-open set in X. Therefore, $f(H) \cap g(H) = \emptyset$ and it follows that $x \notin b\mathcal{I}\,\mathrm{Cl}(E)$. This shows that E is b-$\mathcal{I}$-closed in X. ■

Corollary 3.18. If $f : (X, \tau) \to (Y, \sigma)$ is weakly continuous and $g : (X, \tau) \to (Y, \sigma)$ is contra-b-continuous and Y is Urysohn, then the set $E = \{x \in X : f(x) = g(x)\}$ is b-closed.

Proof. The proof follows from Theorem 3.17, if $\mathcal{I} = \{\emptyset\}$. ■

Definition 3.19. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be a b-$\mathcal{I}$-T_2 space [3] if for each pair of distinct points $x, y \in X$, there exist $U, V \in B\mathcal{I}O(X)$ containing x and y, respectively, such that $U \cap V = \emptyset$.

Theorem 3.20. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a contra-b-$\mathcal{I}$-continuous injective function and Y is a Urysohn space, then $(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-T_2 space.

Proof. Suppose that Y is Urysohn space. By the injectivity of f, it follows that $f(x) \neq f(y)$ for any distinct points x and y in X. Since Y is Urysohn, there exist open sets V and W such that $f(x) \in V$ and $f(y) \in W$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since f is contra-b-$\mathcal{I}$-continuous, there exist b-$\mathcal{I}$-open sets U and G in X containing x and y, respectively, such that $f(U) \subset \mathrm{Cl}(V)$ and $f(G) \subset \mathrm{Cl}(W)$. Hence $U \cap G = \emptyset$. This shows that X is b-$\mathcal{I}$-T_2. ■

Corollary 3.21. ([5], Theorem 2.14) If $f : (X, \tau) \to (Y, \sigma)$ is a contra-b-continuous injective function and Y is a Urysohn space, then (X, τ) is a b-T_2 space.

Proof. The proof follows from Theorem 3.20, if $\mathcal{I} = \{\emptyset\}$. ■

Corollary 3.22. If f is a contra-b-$\mathcal{I}$-continuous injective function of an ideal topological space $(X, \tau, \mathcal{I})$ into a Ultra Hausdoroff (Y, σ), then $(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-T_2 space.

Proof. Let x_1 and x_2 be any distinct points in X. Then since f is injective and Y is Ultra Hausdoroff, $f(x_1) \neq f(x_2)$ and there exist clopen sets V_1 and V_2 of Y such that $f(x_1) \in V_1$ and $f(x_2) \in V_2$ and $V_1 \cap V_2 = \emptyset$. Then $x_i \in f^{-1}(V_i) \in b\mathcal{I}O(X, \tau)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus, $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-T_2. ■

Corollary 3.23. ([5], Corollary 2.2) If f is a contra-b-continuous injective function of a topological space (X, τ) into a Ultra Hausdoroff (Y, σ), then (X, τ) is a b-T_2 space.

Proof. The proof follows from Corollary 3.22, if $\mathcal{I} = \{\emptyset\}$. ■

Definition 3.24. The graph $G(f)$ of a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be contra-b-$\mathcal{I}$-closed in $X \times Y$ if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exist $U \in B\mathcal{I}O(X, x)$ and a closed set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.

Lemma 3.25. The graph $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra-b-$\mathcal{I}$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \backslash G(f)$, there exists $U \in B\mathcal{I}O(X, x)$ and a closed set V of Y containing y such that $f(U) \cap V = \emptyset$.

Proof. The proof is an immediate consequence of Definition 3.24. ■

Theorem 3.26. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a contra-b-$\mathcal{I}$-continuous function and Y is a Urysohn space, then $G(f)$ is contra-b-$\mathcal{I}$-closed in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) \backslash G(f)$. Then $y \neq f(x)$ and there exist open sets V, W of Y such that $f(x) \in V$, $y \in W$ and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. Since f is contra-b-$\mathcal{I}$-continuous, there exists $U \in B\mathcal{I}O(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Therefore, we obtain $f(U) \cap \mathrm{Cl}(W) = \emptyset$. This shows that $G(f)$ is contra-b-$\mathcal{I}$-closed.

The following example shows that the condition Urysohn on the space Y in Theorem 3.26 cannot be dropped. ■

Example 3.27. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{c\}, \{b, c\}, \{a, c\}, X\}$,$\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Clearly, (X, σ) is not a Urysohn space. Also the identity function $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ is contra-b-$\mathcal{I}$-continuous but not contra-b-$\mathcal{I}$-closed.

Definition 3.28. The graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be contra-γ-closed [14] in $X \times Y$ if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in bO(X, x)$ and a closed set V of Y containing y such that $(U \times V) \cap G(f) = \emptyset$.

Corollary 3.29. ([14], Theorem 3.11) If $f : (X, \tau) \to (Y, \sigma)$ is a contra-b-continuous function and Y is a Urysohn space, then $G(f)$ is contra-b-closed in $X \times Y$.

Proof. The proof follows from Theorem 3.26, if $\mathcal{I} = \{\emptyset\}$. ■

Theorem 3.30. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a b-$\mathcal{I}$-continuous function and (Y, σ) is T_1, then $G(f)$ is contra-b-$\mathcal{I}$-closed.

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$. Then $y \neq f(x)$ and there exists an open set V in Y such that $f(x) \in V$ and $y \notin V$. Since f is b-$\mathcal{I}$-continuous, there exists $U \in B\mathcal{I}O(X, x)$ such that $f(U) \subset \mathrm{Cl}(V)$. Therefore, $f(U) \cap (Y\backslash V) = \emptyset$ and $(Y\backslash V)$ is a closed set of Y containing y. This shows that $G(f)$ is contra-b-$\mathcal{I}$-closed.

The following example shows that the condition T_1 on the space Y in Theorem 3.30 cannot be dropped. ■

Example 3.31. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Clearly, (X, τ) is not a T_1 space. Also the identity function $f : (X, \tau, \mathcal{I}) \to (X, \tau)$ is b-$\mathcal{I}$-continuous but not contra-b-$\mathcal{I}$-closed.

Corollary 3.32. ([14],Theorem 3.12) If $f : (X, \tau) \to (Y, \sigma)$ is a γ-continuous function and (Y, σ) is T_2, then $G(f)$ is contra-γ-closed.

Proof. The proof follows from Theorem 3.30, if $\mathcal{I} = \{\emptyset\}$. ■

Definition 3.33. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-connected [1] if X cannot be expressed as the union of two nonempty disjoint b-$\mathcal{I}$-open sets.

Theorem 3.34. A contra-b-$\mathcal{I}$-continuous image of a b-$\mathcal{I}$-connected space is connected.

Proof. The proof is clear.

Recall that a topological space (X, τ) is said to be a γ-connected [8] space if X is not the union of two disjoint nonempty b-open sets of X. ■

Corollary 3.35. ([14], Theorem 4.2) *A* contra-γ-continuous image of a γ-connected space is connected.

Proof. The proof follows from Theorem 3.34, if $\mathcal{I} = \{\emptyset\}$. ■

Theorem 3.36. Let $(X, \tau, \mathcal{I})$ be a b-$\mathcal{I}$-connected space and Y be a T_1 space. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a contra-b-$\mathcal{I}$-continuous function, then it is a constant function.

Proof. Since Y is a T_1 space, $\Lambda = \{f^{-1}(\{y\}) : y \in Y\}$ is a partition of X by b-$\mathcal{I}$-open sets. If $|\Lambda| \geq 2$, then X is the union of two nonempty b-$\mathcal{I}$-open sets. Since $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-connected, $|\Lambda| = 1$. Hence, f is a constant function. ■

Corollary 3.37. ([14],Theorem 4.3) Let (X, τ) be a γ-connected space and Y be a T_1 space. If $f : (X, \tau) \to (Y, \sigma)$ is a contra-γ-continuous function, then it is a constant function.

Proof. The proof follows from Theorem 3.36, if $\mathcal{I} = \{\emptyset\}$. ■

Theorem 3.38. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function and let $\{U_\alpha : \alpha \in \triangle\}$ be a cover of X by open sets. If the restriction function $f_{|U_\alpha} : (U_\alpha, \tau_{|U_\alpha}, \mathcal{I}_{|U_\alpha})$ is contra-b-$\mathcal{I}$-continuous for each $\alpha \in \triangle$, then f is contra-b-$\mathcal{I}$-continuous.

Proof. It follows from Theorem 3.15 of [1]. ■

Definition 3.39. A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be b-$\mathcal{I}$-irresolute if $f^{-1}(V)$ is b-$\mathcal{I}$-open in X for every b-$\mathcal{J}$-open set V of Y.

Theorem 3.40. For the functions $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \to (Z, \eta)$ the following hold:

1. $g \circ f$ is b-$\mathcal{I}$-continuous, if f is contra-b-$\mathcal{I}$-continuous and g is contra-continuous.
2. $g \circ f$ is contra-b-$\mathcal{I}$-continuous, if f is contra-b-$\mathcal{I}$-continuous and g is continuous.
3. $g \circ f$ is b-$\mathcal{I}$-continuous, if f is b-$\mathcal{I}$-irresolute and g is contra-b-$\mathcal{J}$-continuous.

Remark 3.41. The following example shows that composition of any contra-b-$\mathcal{I}$-continuous functions need not be contra-b-$\mathcal{I}$-continuous function in general.

Example 3.42. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{c\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the identity functions $f : (X, \tau_1, \mathcal{I}) \to (X, \tau_2, \mathcal{I})$ and $g : (X, \tau_2, \mathcal{I}) \to (X, \tau_1, \mathcal{I})$ are contra-b-$\mathcal{I}$-continuous but their composition g ∘ f is not contra-b-$\mathcal{I}$-continuous.

4. b-$\mathcal{I}$-Normal spaces

Definition 4.1. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be:

1. b-$\mathcal{I}$-normal if each pair of nonempty disjoint closed sets can be separated by disjoint b-$\mathcal{I}$-open sets.
2. $\mathcal{I}$-normal [12] if each pair of nonempty disjoint closed sets can be separated by disjoint b-$\mathcal{I}$-open sets.

Proposition 4.2. If $\mathcal{I}$ and $\mathcal{J}$ are ideals on X having $\mathcal{I} \subseteq \mathcal{J}$. Then $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-normal if $(X, \tau, \mathcal{J})$ is b-$\mathcal{J}$-normal.

Proof. It follows from the fact that $B\mathcal{I}O(X, \tau) \subseteq B\mathcal{J}O(X, \tau)$ when $\mathcal{I} \subseteq \mathcal{J}$. ■

Proposition 4.3. Every open subspace of a b-$\mathcal{I}$-normal space is b-$\mathcal{I}$-normal.

Proof. It follows from Theorem 3.15 of [1]. ■

Definition 4.4. A topological space $(X, \tau)$is said to be:

1. ultra normal [17] if each pair of nonempty disjoint closed sets can be separated by disjoint clopen sets.
2. b-normal [16] if each pair of nonempty disjoint closed sets can be separated by disjoint b-open sets.

Proposition 4.5.

1. Every $\mathcal{I}$-normal space is b-$\mathcal{I}$-normal.
2. Every b-$\mathcal{I}$-normal space is b-normal.
3. Every normal space is b-$\mathcal{I}$-normal.
4. Every ultra normal space is b-$\mathcal{I}$-normal.

Proof. The proof is clear.
The following examples show that the converses of the above Proposition is not true, in general. ■

Example 4.6. Let $X = \{a, b, c, d\}$,$\tau = \{\emptyset, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$ and $\mathcal{I} = \{\emptyset\}$. Then the ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-normal but none of normal and ultra normal.

Example 4.7. Let $X = \{a, b, c, d\}$,$\tau = \{\emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then the ideal topological space $(X, \tau, \mathcal{I})$ is b-normal but not b-$\mathcal{I}$-normal.

Example 4.8. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then the ideal topological space $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-normal but not $\mathcal{I}$-normal.

Theorem 4.9. For an ideal topological space $(X, \tau, \mathcal{I})$, the following statements are equivalent:

1. $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-normal;
2. for every pair of open sets U and V whose union is X, there exists b-$\mathcal{I}$-closed sets A and B such that $A \subset U$, $B \subset V$ and $A \cup B = X$;

3. for every closed set H and every open set K containing H, there exists a b-$\mathcal{I}$-open set U such that $H \subset U \subset b\mathcal{I}\,\mathrm{Cl}(U) \subset K$.

Proof.

(1) $\Rightarrow$ (2): Let U and V be a pair of open sets in a b-$\mathcal{I}$-normal space $(X, \tau, \mathcal{I})$ such that $X = U \cup V$. Then $X\backslash U$,$X\backslash V$ are disjoint closed sets. Since $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-normal, there exist disjoint b-$\mathcal{I}$-open sets U_1 and V_1 such that $X\backslash U \subset U_1$, and $X\backslash V \subset V_1$. Let $A = X\backslash U_1$ and $B = X\backslash V_1$. Then A and B are b-$\mathcal{I}$-closed sets such that $A \subset U$,$B \subset V$ and $A \cup B = X$.

(2) $\Rightarrow$ (3): Let H be a closed set and K be an open set containing H. Then $X\backslash H$ and K are open sets whose union is X. Then by (2), there exist b-$\mathcal{I}$-closed sets M_1 and M_2 such that $M_1 \subset X\backslash H$ and $M_2 \subset X\backslash K$ and $M_1 \cup M_2 = X$. Then $H \subset X\backslash M_1$, $X\backslash K \subset X\backslash M_2$ and $(X\backslash M_1) \cap (X\backslash M_2) = \emptyset$. Then U and V are disjoint b-$\mathcal{I}$-open sets such that $H \subset U \subset X\backslash V \subset K$. As $X\backslash V$ is b-$\mathcal{I}$-closed, we have $b\mathcal{I}\,\mathrm{Cl}(U) \subset X\backslash V$ and $H \subset U \subset b\mathcal{I}\,\mathrm{Cl}(U) \subset K$.

(3) $\Rightarrow$ (1): Let H_1 and H_2 be any two disjoint closed sets of X. Put $K = X\backslash H_2$, then $H_2 \cap K = \emptyset$. By (iii), there exists a b-$\mathcal{I}$-open set U of X such that $H_1 \subset U \subset b\mathcal{I}\,\mathrm{Cl}(U) \subset K$. It follows that $H_2 \subset X\backslash b\mathcal{I}\,\mathrm{Cl}(U) = V$,say, then V is a b-$\mathcal{I}$-open set and $U \cap V = \emptyset$. Therefore, $(X, \tau, \mathcal{I})$ is b-$\mathcal{I}$-normal. ■

Corollary 4.10. ([16], Theorem 4.4) For a topological space (X, τ), the following statements are equivalent:

1. (X, τ) is b-normal;
2. for every pair of open sets U and V whose union is X, there exists b-closed sets A and B such that $A \subset U$,$B \subset V$ and $A \cup B = X$;
3. for every closed set H and every open set K containing H, there exists a b-open set U such that $H \subset U \subset b\,\mathrm{Cl}(U) \subset K$.

Proof. The proof follows from Theorem 4.9, if $\mathcal{I} = \{\emptyset\}$. ■

Theorem 4.11. If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a contra-b-$\mathcal{I}$-continuous closed injective function and Y is a ultra normal space, then $(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-normal space.

Proof. Let F_1 and F_2 be a disjoint closed subsets of X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y. Since Y is ultra normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets V_1 and V_2, respectively. Hence $F_i \subset f^{-1}(V_i)$, $f^{-1}(V_i) \in B\mathcal{I}O(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$. Thus, $(X, \tau, \mathcal{I})$ is a b-$\mathcal{I}$-normal. ■

Corollary 4.12. ([5], Theorem 3.1) If $f : (X, \tau) \to (Y, \sigma)$ is a contra-b-continuous closed injective function and Y is a ultra normal space, then (X, τ) is a b-normal space.

Proof. The proof follows from Theorem 4.11, if $\mathcal{I} = \{\emptyset\}$. ∎

Definition 4.13. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be b-$\mathcal{I}$-closed compact if for every b-$\mathcal{I}$-closed cover $\{W_i : i \in \triangledown\}$, there exists a finite subset $\triangledown_\circ$ of $\triangledown$ such that $X \setminus \cup \{U_i : i \in \triangledown_\circ\} \in \mathcal{I}$.

Lemma 4.14. [15] For any function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$, $f(\mathcal{I})$ is an ideal on Y.

Theorem 4.15. If a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is contra-b-$\mathcal{I}$-continuous and the set A is b-$\mathcal{I}$-closed compact relative to X, then $f(A)$ is $f(\mathcal{I})$-compact in Y.

Proof. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a contra-b-$\mathcal{I}$-continuous surjection and $\{V_i : i \in \triangledown\}$ be an open cover of Y. Then $\{f^{-1}(V_i) : i \in \triangledown_\circ\}$ is a b-$\mathcal{I}$-closed cover of X. From the assumption, there exists a finite subset $\triangledown_\circ$ of $\triangledown$ such that $X \setminus \cup \{f^{-1}(V_i) : i \in \triangledown_\circ\} \in \mathcal{I}$. Therefore, $Y \setminus \cup \{V_i : i \in \triangledown_\circ\} \in f(\mathcal{I})$ which shows that Y is $f(\mathcal{I})$-compact. ∎

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