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q-analogues of Boole polynomials

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Abstract

Recently, Boole polynomials have been studied by Kim and Kim over the p-adic number field. In this paper, we consider a q-extension of Boole polynomials by using the fermionic p-adic integrals on $\mathbb{Z}_p$ and give some new identities related to those polynomials.

AMS subject classification:
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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = 1/p$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [11]).} \tag{1.1}$$

From (1), we have

$$I_{-1}(f_n) = 2\sum_{x=0}^{n-1}(-1)^{n-1-a} f(a) + (-1)^n I_{-1}(f). \tag{1.2}$$

Let us assume that q is an indeterminate in $\mathbb{C}_p$ with $|1-q|_p < p^{-1/(p-1)}$. The Stirling number of the first kind is defined by

$$(x)_n = \sum_{\ell=0}^{n} S_1(n,\ell)x^\ell, \ (n \geq 0), \tag{1.3}$$

and the Stirling number of the second kind is given by

$$x^n = \sum_{\ell=0}^{n} S_2(n,\ell)(x)_\ell, \ (n \geq 0), \text{ (see [9, 16]).} \tag{1.4}$$

The Boole polynomials are defined by the generating function to be

$$\sum_{n=0}^{\infty} Bl_n(x|\lambda)\frac{t^n}{n!} = \frac{1}{1+(1+t)^\lambda}(1+t)^x, \text{ (see [9, 16]).} \tag{1.5}$$

In [9], Kim and Kim gave a Witt-type formula for $Bl_n(x|\lambda)$ over the p-adic number field as follows:

$$\int_{\mathbb{Z}_p} (x+\lambda y)_n d\mu_{-1}(y) = 2Bl_n(x|\lambda), \ (n \geq 0), \tag{1.6}$$

where $\lambda \in \mathbb{Z}_p$ and $(x)_n = x(x-1)\cdots(x-n+1)$.

Let us define the q-product of x as follows:

$$(x)_{n,q} = x(x-q)(x-2q)\cdots(x-(n-1)q), \ (n \geq 0). \tag{1.7}$$

As is known, the Euler polynomials are defined by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \text{ (see [1 − 20]).} \tag{1.8}$$

When $x = 0$, $E_n = E_n(0)$ are called the Euler numbers. From (2), we have

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.9}$$

In this paper, we consider a q-extension of Boole polynomials by using the fermionic p-adic integrals on $\mathbb{Z}_p$ and give some new identities of those polynomials.

2. Boole polynomials with q-parameter

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-1/(p-1)}|q|_p$ and $\lambda \in \mathbb{Z}_p$. Now, we consider the Boole polynomials with q-parameter as follows:

$$Bl_{n,q}(x|\lambda) = \int_{\mathbb{Z}_p} (x + \lambda y)_{n,q}\, d\mu_{-1}(y), \ (n \geq 0). \tag{2.1}$$

Thus, by (10), we get

$$\begin{aligned} Bl_{n,q}(x|\lambda) &= \sum_{\ell=0}^{n} S_1(n,\ell) q^{n-\ell} \lambda^{\ell} \int_{\mathbb{Z}_p} \left(\frac{x}{\lambda} + y\right)^{\ell} d\mu_{-1}(y) \\ &= \sum_{\ell=0}^{n} S_1(n,\ell) q^{n-\ell} \lambda^{\ell} E_{\ell}\left(\frac{x}{\lambda}\right). \end{aligned} \tag{2.2}$$

From (2) and (10), we can derive the generating function of $Bl_{n,q}(x|\lambda)$ as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} Bl_{n,q}(x|\lambda) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x + \lambda y)_{n,q} d\mu_{-1}(y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p} \binom{\frac{x+\lambda y}{q}}{n} d\mu_{-1}(y) t^n \\ &= \int_{\mathbb{Z}_p} (1 + qt)^{\frac{x+\lambda y}{q}} d\mu_{-1}(y) \\ &= (1 + qt)^{\frac{x}{q}} \left(\frac{2}{(1 + qt)^{\lambda/q} + 1} \right). \end{aligned} \tag{2.3}$$

Therefore, by (12), we obtain the following theorem.

Theorem 2.1. Let $F(t, x|\lambda) = \sum_{n=0}^{\infty} Bl_{n,q}(x|\lambda) \frac{t^n}{n!}$. Then we have

$$F(t, x|\lambda) = \left(\frac{2}{(1 + qt)^{\lambda/q} + 1} \right) (1 + qt)^{x/q}.$$

By replacing t by $(e^t-1)/q$ in (12), we get

$$\begin{aligned}\sum_{n=0}^{\infty} Bl_{n,q}(x|\lambda)q^{-n}\frac{(e^t-1)^n}{n!} =&\frac{2}{e^{\lambda t/q}+1}e^{\frac{x}{q}t}\\ =&\sum_{n=0}^{\infty} E_n\left(\frac{x}{\lambda}\right)\left(\frac{\lambda}{q}\right)^n\frac{t^n}{n!}\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} Bl_{n,q}(x|\lambda)q^{-n}\frac{(e^t-1)^n}{n!} =&\sum_{n=0}^{\infty} Bl_{n,q}(x|\lambda)q^{-n}\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!}\\ =&\sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m} Bl_{n,q}(x|\lambda)\frac{S_2(m,n)}{q^n}\right\}\frac{t^m}{m!}.\end{aligned} \tag{2.5}$$

Therefore, by (13) and (14), we obtain the following theorem.

Theorem 2.2. For $m \geq 0$, we have

$$\lambda^m E_m\left(\frac{x}{\lambda}\right) = \sum_{n=0}^{m} Bl_{n,q}(x|\lambda)q^{m-n}S_2(m,n),$$

and

$$Bl_{m,q}(x|\lambda) = \sum_{\ell=0}^{m} S_1(m,\ell)q^{m-\ell}\lambda^{\ell}E_{\ell}\left(\frac{x}{\lambda}\right).$$

Note that $\lim_{q\to 1} Bl_{n,q}(x|\lambda) = 2Bl_n(x|\lambda),\ (n \geq 0)$. When $x=0$, $Bl_{n,q}(\lambda) = Bl_{n,q}(0|\lambda)$ are called the q-Boole numbers.

Now, we consider the q-Boole polynomials of the second kind as follows:

$$\widehat{Bl}_{n,q}(x|\lambda) = \int_{\mathbb{Z}_p} (-\lambda y + x)_{n,q}d\mu_{-1}(y),\ (n \geq 0). \tag{2.6}$$

Thus, by (15), we get

$$\begin{aligned}\widehat{Bl}_{n,q}(x|\lambda) =&q^n\int_{\mathbb{Z}_p}\left(\frac{-\lambda y+x}{q}\right)_n d\mu_{-1}(y)\\ =&q^n\int_{\mathbb{Z}_p}\sum_{\ell=0}^{n}\frac{\lambda^{\ell}S_1(n,\ell)}{q^{\ell}}(-1)^{\ell}\left(y-\frac{x}{\lambda}\right)^{\ell}d\mu_{-1}(y)\\ =&\sum_{\ell=0}^{n} S_1(n,\ell)q^{n-\ell}\lambda^{\ell}(-1)^{\ell}E_{\ell}\left(-\frac{x}{\lambda}\right).\end{aligned} \tag{2.7}$$

From (8), we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_n\left(-\frac{x}{\lambda}\right)\frac{t^n}{n!} =& \frac{2}{e^t+1}e^{(-\frac{x}{\lambda})t}\\ =& \frac{2}{1+e^{-t}}e^{-(1+\frac{x}{\lambda})t}\\ =& \sum_{n=0}^{\infty}(-1)^n E_n\left(1+\frac{x}{\lambda}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

By (16) and (17), we get

$$\widehat{Bl}_{n,q}(x|\lambda) = \sum_{\ell=0}^{n} \lambda^{\ell}|S_1(n,\ell)|q^{n-\ell}E_{\ell}\left(1+\frac{x}{\lambda}\right). \tag{2.9}$$

From (15), we can derive the generating function of the Boole polynomials of the second kind as follows:

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{Bl}_{n,q}(x|\lambda)\frac{t^n}{n!} =& \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p}\binom{\frac{-\lambda y+x}{q}}{n} d\mu_{-1}(y)t^n\\ =& \int_{\mathbb{Z}_p}(1+qt)^{\frac{-\lambda y+x}{q}}d\mu_{-1}(y)\\ =& (1+qt)^{\frac{x}{q}}\int_{\mathbb{Z}_p}(1+qt)^{-\frac{\lambda y}{q}}d\mu_{-1}(y)\\ =& (1+qt)^{\frac{x+\lambda}{q}}\frac{2}{(1+qt)^{\lambda/q}+1}.\end{aligned} \tag{2.10}$$

By replacing t by $(e^t-1)/q$ in (19), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{Bl}_{n,q}(x|\lambda)\frac{1}{q^n}\frac{(e^t-1)^n}{n!} =& e^{\frac{1}{q}(x+\lambda)t}\frac{2}{e^{\frac{\lambda}{q}t}+1}\\ =& \sum_{n=0}^{\infty}E_n\left(\frac{x+\lambda}{\lambda}\right)\frac{\lambda^n}{q^n}\frac{t^n}{n!}\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{Bl}_{n,q}(x|\lambda)q^{-n}\frac{(e^t-1)^n}{n!} =& \sum_{n=0}^{\infty}\widehat{Bl}_{n,q}(x|\lambda)q^{-n}\sum_{m=n}^{\infty}S_2(m,n)\frac{t^m}{m!}\\ =& \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{Bl}_{n,q}(x|\lambda)\frac{S_2(m,n)}{q^n}\right)\frac{t^m}{m!}.\end{aligned} \tag{2.12}$$

Therefore, by (18), (20) and (21), we obtain the following theorem.

Theorem 2.3. For $m \geq 0$, we have

$$\sum_{n=0}^{m} \widehat{Bl}_{n,q}(x|\lambda) q^{m-n} S_2(m,n) = \lambda^m E_m\left(1+\frac{x}{\lambda}\right)$$

and

$$\widehat{Bl}_{m,q}(x|\lambda) = \sum_{\ell=0}^{m} S_1(m,\ell) q^{m-\ell}\lambda^{\ell} E_{\ell}\left(1+\frac{x}{\lambda}\right).$$

For $\alpha \in \mathbb{N}$, let us consider q-Boole polynomials of the first kind with order α as follows:

$$\begin{aligned} Bl_{n,q}^{(\alpha)}(x|\lambda) =& \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (\lambda x_1+\cdots+\lambda x_\alpha + x)_{n,q}\, d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_\alpha) \\ =& q^n \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \left(\frac{\lambda x_1+\cdots+\lambda x_\alpha + x}{q}\right)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_\alpha) \\ =& q^n \sum_{\ell=0}^{n} S_1(n,\ell)\frac{1}{q^\ell}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (\lambda x_1+\cdots+\lambda x_\alpha + x)^{\ell}\, d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_\alpha) \\ =& \sum_{\ell=0}^{n} S_1(n,\ell) q^{n-\ell}\lambda^{\ell} E_{\ell}^{(\alpha)}\left(\frac{x}{\lambda}\right), \end{aligned} \tag{2.13}$$

where $E_n^{(\alpha)}(x)$ are the Euler polynomials of order α which are defined by

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}.$$

From (22), we note that the generating function of $Bl_{n,q}^{(\alpha)}(x|\lambda)$ are given by

$$\begin{aligned} \sum_{n=0}^{\infty} Bl_{n,q}^{(\alpha)}(x|\lambda)\frac{t^n}{n!} =& \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \binom{\frac{\lambda x_1+\cdots+\lambda x_\alpha + x}{q}}{n} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_\alpha) t^n \\ =& \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (1+qt)^{\frac{\lambda x_1+\cdots+\lambda x_\alpha + x}{q}} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_\alpha) \\ =& (1+qt)^{\frac{x}{q}}\left(\frac{2}{(1+qt)^{\frac{\lambda}{q}}+1}\right)^{\alpha}. \end{aligned} \tag{2.14}$$

By replacing t by $(e^t - 1)/q$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} Bl_{n,q}^{(\alpha)}(x|\lambda)\frac{1}{q^n}\frac{(e^t-1)^n}{n!} &= e^{\frac{x}{q}t}\left(\frac{2}{e^{\frac{\lambda}{q}t}+1}\right)^{\alpha} \\ &= \sum_{n=0}^{\infty} E_n^{(\alpha)}\left(\frac{x}{\lambda}\right)\frac{\lambda^n}{q^n}t^n\end{aligned} \tag{2.15}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} Bl_{n,q}^{(\alpha)}(x|\lambda)\frac{1}{q^n}\frac{(e^t-1)^n}{n!} &= \sum_{n=0}^{\infty} Bl_{n,q}^{(\alpha)}(x|\lambda)\frac{1}{q^n}\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} Bl_{n,q}^{(\alpha)}(x|\lambda)\frac{S_2(m,n)}{q^n}\right)\frac{t^m}{m!}.\end{aligned} \tag{2.16}$$

Therefore, by (24) and (25), we obtain the following theorem.

Theorem 2.4. For $m \geq 0$, we have

$$\lambda^m E_m^{(\alpha)}\left(\frac{x}{\lambda}\right) = \sum_{n=0}^{m} q^{m-n} Bl_{n,q}^{(\alpha)}(x|\lambda) S_2(m,n)$$

and

$$Bl_{m,q}^{(\alpha)}(x|\lambda) = \sum_{\ell=0}^{m} S_1(m,\ell) q^{m-\ell}\lambda^{\ell} E_{\ell}^{(\alpha)}\left(\frac{x}{\lambda}\right).$$

We consider the q-Boole polynomials of the second kind with order α as follows:

$$\widehat{Bl}_{n,q}^{(\alpha)}(x|\lambda) = \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-\lambda x_1 - \cdots - \lambda x_\alpha + x)_{n,q}\, d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_\alpha). \tag{2.17}$$

Then, by (26), we get

$$\begin{aligned}\widehat{Bl}_{n,q}^{(\alpha)}(x|\lambda) &= q^n \sum_{\ell=0}^{n} S_1(n,\ell)(-1)^{\ell}\left(\frac{\lambda}{q}\right)^{\ell}\int_{\mathbb{Z}_p}\cdots \\ &\quad \int_{\mathbb{Z}_p}\left(x_1+\cdots+x_\alpha-\frac{x}{\lambda}\right)^{\ell} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_\alpha) \\ &= \sum_{\ell=0}^{n} S_1(n,\ell)(-1)^{\ell} q^{n-\ell}\lambda^{\ell} E_{\ell}^{(\alpha)}\left(-\frac{x}{\lambda}\right).\end{aligned} \tag{2.18}$$

From the definition of the higher-order Euler polynomials, we note that

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^{(\alpha)}\left(-\frac{x}{\lambda}\right)\frac{t^n}{n!} &= \left(\frac{2}{e^t+1}\right)^{\alpha} e^{-\frac{x}{\lambda}t} \\ &= \left(\frac{2}{1+e^{-t}}\right)^{\alpha} e^{-(\frac{x}{\lambda}+\alpha)t} \\ &= \sum_{n=0}^{\infty}(-1)^n E_n^{(\alpha)}\left(\frac{x}{\lambda}+\alpha\right)\frac{t^n}{n!}.\end{aligned} \tag{2.19}$$

Thus, by (27) and (28), we get

$$\widehat{Bl}_{n,q}^{(\alpha)}(x|\lambda) = \sum_{\ell=0}^{n} S_1(n,\ell)q^{n-\ell}\lambda^{\ell}E_{\ell}^{(\alpha)}\left(\frac{x}{\lambda}+\alpha\right). \tag{2.20}$$

From (26), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{Bl}_{n,q}^{(\alpha)}(x|\lambda)\frac{t^n}{n!} =& \sum_{n=0}^{\infty} q^n \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{\frac{-\lambda x_1+\cdots-\lambda x_\alpha+x}{q}}{n} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_\alpha)t^n \\ =& \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+qt)^{\frac{x-\lambda x_1-\cdots-\lambda x_\alpha}{q}} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_\alpha) \\ =& (1+qt)^{\frac{x+\alpha}{q}}\left(\frac{2}{(1+qt)^{\frac{\lambda}{q}}+1}\right)^{\alpha}.\end{aligned} \tag{2.21}$$

By replacing t by $(e^t-1)/q$, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{Bl}_{n,q}^{(\alpha)}(x|\lambda)\frac{(e^t-1)^n}{n!q^n} =& e^{\frac{x+\alpha}{q}t}\left(\frac{2}{e^{\frac{\lambda}{q}t}+1}\right)^{\alpha} \\ =& \sum_{n=0}^{\infty} E_n^{(\alpha)}\left(\frac{x+\alpha}{\lambda}\right)\left(\frac{\lambda}{q}\right)^n\frac{t^n}{n!}.\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{Bl}_{n,q}^{(\alpha)}(x|\lambda)\frac{1}{q^n}\frac{1}{n!}(e^t-1)^n =& \sum_{n=0}^{\infty}\widehat{Bl}_{n,q}^{(\alpha)}(x|\lambda)\frac{1}{q^n}\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} \\ =& \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\widehat{Bl}_{n,q}^{(\alpha)}(x|\lambda)\frac{S_2(m,n)}{q^n}\right)\frac{t^m}{m!}.\end{aligned} \tag{2.23}$$

Therefore, by (31) and (32), we obtain the following theorem.

Theorem 2.5. For $m \geq 0$, we have

$$\lambda^m E_m^{(\alpha)}\left(\frac{x+\alpha}{\lambda}\right) = \sum_{n=0}^{m} q^{m-n} \widehat{Bl}_{n,q}^{(\alpha)}(x|\lambda) S_2(m,n) \tag{2.24}$$

and

$$\widehat{Bl}_{m,q}^{(\alpha)}(x|\lambda) = \sum_{\ell=0}^{m} S_1(m,\ell) q^{m-\ell} \lambda^\ell E_\ell^{(\alpha)}\left(\frac{x+\alpha}{\lambda}\right). \tag{2.25}$$

Remark 2.6. When $x = 0$, $\widehat{Bl}_{n,q}(\lambda) = \widehat{Bl}_{n,q}(0|\lambda)$ are called the q-Boole numbers of the second kind.

Now, we observe that

$$\begin{aligned}
\frac{\widehat{Bl}_{n,q}(\lambda)}{n!} &= \frac{1}{n!}\int_{\mathbb{Z}_p} (-\lambda y)_{n,q} d\mu_{-1}(y) \\
&= q^n \int_{\mathbb{Z}_p} \binom{\frac{-\lambda y}{q}}{n} d\mu_{-1}(y) \\
&= (-1)^n q^n \int_{\mathbb{Z}_p} \binom{\frac{\lambda y}{q}+n-1}{n} d\mu_{-1}(y) \\
&= (-1)^n q^n \sum_{\ell=0}^{n} \binom{n-1}{\ell-1} \frac{1}{\ell!} \int_{\mathbb{Z}_p} \left(\frac{\lambda y}{q}\right)_\ell d\mu_{-1}(y) \\
&= (-1)^n q^n \sum_{\ell=0}^{n} \binom{n-1}{\ell-1} \frac{Bl_{\ell,q}(\lambda)}{\ell! q^\ell}.
\end{aligned} \tag{2.26}$$

Therefore, by (35), we obtain the following theorem.

Theorem 2.7. For $n \geq 0$, we have

$$\frac{(-1)^n}{q^n} \frac{\widehat{Bl}_{n,q}(\lambda)}{n!} = \sum_{\ell=0}^{n} \binom{n-1}{\ell-1} \frac{Bl_{\ell,q}(\lambda)}{\ell! q^\ell}. \tag{2.27}$$

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