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A note on Barnes-type Daehee polynomials associated with p-adic invariant integral on $\mathbb{Z}_p$

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Abstract

In this paper, we investigate some interesting properties of Barnes-type Daehee polynomials and consider Witt-type formula for the Barnes-type Daehee numbers and polynomials. Finally, we derive some new interesting identities and properties of those polynomials from the Witt-type formula which are related to the Daehee polynomials.

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1. Introduction

Let p be an odd prime number. $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$.

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If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$\begin{aligned} I_0(f) = \int_{\mathbb{Z}_p} f(x) d\mu_0(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_0(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x) \end{aligned} \tag{1.1}$$

From (1.1), we have

$$I_0(f_1) = I_0(f) + f'(0), \quad \text{(see [2-7])}. \tag{1.2}$$

where $f_1(x) = f(x+1)$.

By using iterative method, we get

$$I_0(f_n) = I_0(f) + \sum_{i=0}^{n-1} f'(i), \quad \text{(see [1,8,9])}. \tag{1.3}$$

where $f_n(x) = f(x+n)$, $(n \in \mathbb{N})$.

We define the Daehee polynomials are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t}\right)(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \quad \text{(see [10, 12, 13, 14])}. \tag{1.4}$$

The *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \ (n \geq 0), \quad \text{(see [1,11,15])}.$$

and the *Stirling numbers of the second kind* is defined by

$$(e^t-1)^n = n! \sum_{l=n}^{\infty} S_2(l,n) \frac{t^l}{l!}.$$

In this paper, we investigate some properties of Barnes-type Daehee polynomials and consider Witt-type formulas the Barnes-type Daehee numbers and polynomials. Finally, we derive some new identities of those polynomials from the Witt-type formulas which are related to Barnes-type Daehee polynomials.

2. Barnes-type Daehee polynomials

Let $a_1, a_2, \cdots, a_r \neq 0 \in \mathbb{C}_p$. Then, by (1.2), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{(a_1x_1+a_2x_2+\cdots+a_rx_r+x)} d\mu_0(x_1)\cdots d\mu_0(x_r) \\
&= \left(\prod_{i=1}^{r} a_i\right)\left(\frac{(log(1+t))^r}{((1+t)^{a_1}-1)((1+t)^{a_2}-1)\cdots((1+t)^{a_r}-1)}\right)(1+t)^x \\
&= \left(\prod_{i=1}^{r} a_i\right)\sum_{n=0}^{\infty} D_n(x|a_1, a_2, \cdots, a_r)\frac{t^n}{n!}.
\end{aligned} \tag{2.1}$$

When $x = 0$, $D_n(0|a_1, a_2, \cdots, a_r) = D_n(a_1, a_2, \cdots, a_r)$ are called the Barnes-type Daehee numbers.

From (2.1), we obtain the following Witt's formula for the Barnes-type Daehee polynomials.

Theorem 2.1. For $a_1, a_2, \cdots, a_r \neq 0 \in \mathbb{C}_p$, we have

$$D_n(x|a_1, \cdots, a_r) = \left(\prod_{i=1}^{r} a_i\right)^{-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1x_1+\cdots+a_rx_r+x)^n d\mu_0(x_1)\cdots d\mu_0(x_r).$$

Note that

$$\begin{aligned}
(a_1x_1+\cdots+a_rx_r)^n &= \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \cdots, l_r} a_1^{l_1}x_1^{l_1}\cdots a_r^{l_r}x_r^{l_r} \\
&= \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \cdots, l_r}\left(\prod_{i=1}^{r} a_i^{l_i}\right) x_1^{l_1}\cdots x_r^{l_r}.
\end{aligned} \tag{2.2}$$

By (2.2) and Theorem 2.1, we obtain the following corollary.

Corollary 2.2. For $n \geq 2$, we have

$$D_n(a_1, \cdots, a_r) = \sum_{l_1+\cdots+l_r=n} \binom{n}{l_1, \cdots, l_r}\left(\prod_{i=1}^{r} a_i^{l_i-1}\right) D_{l_1}\cdots D_{l_r},$$

where $D_n = D_n(1)$ is the nth Daehee number.

From (1.1), we can easily derive the following integral equation:

$$\int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \frac{1}{d}\sum_{a=0}^{d-1}\int_{\mathbb{Z}_p} f(a+dx)d\mu_0(x), \tag{2.3}$$

where $d \in \mathbb{N}$.

By (2.3), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{(a_1x_1+a_2x_2+\cdots+a_rx_r+x)} d\mu_0(x) \\
&= \frac{1}{d^r} \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{(l_1a_1+\cdots+l_ra_r+a_1dx_1+\cdots+a_rdx_r+x)} d\mu_0(x_1)\cdots d\mu_0(x_r) \\
&= \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} \sum_{n=0}^{\infty} \frac{d^n}{d^r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(\frac{l_1a_1+\cdots+l_ra_r}{d} + a_1x_1+\cdots+a_rx_r+\frac{x}{d} \right)^n \\
&\times d\mu_0(x_1)\cdots d\mu_0(x_r) \frac{t^n}{n!}.
\end{aligned} \tag{2.4}$$

By Theorem 2.1 and (2.4), we get

$$D_n(x|a_1,\cdots,a_r) = d^{n-r} \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} D_n\left(\frac{l_1a_1+\cdots+l_ra_r+x}{d} \middle| a_1,\cdots,a_r \right). \tag{2.5}$$

Therefore, by (2.5), we obtain the following distribution relation for a Barnes-type Daehee polynomial.

Theorem 2.3. For $n \geq 0$, we have

$$D_n(x|a_1,\cdots,a_r) = d^{n-r} \sum_{l_1=0}^{d-1} \cdots \sum_{l_r=0}^{d-1} D_n\left(\frac{l_1a_1+\cdots+l_ra_r+x}{d} \middle| a_1,\cdots,a_r \right).$$

From (1.3), we note that

$$\int_{\mathbb{Z}_p} (1+t)^{a_1(x_1+n)} d\mu_0(x_1) - \int_{\mathbb{Z}_p} (1+t)^{a_1x_1} d\mu_0(x_1) = a_1t \sum_{l=0}^{n-1} (1+t)^{a_1l}. \tag{2.6}$$

By (2.6), we get

$$\int_{\mathbb{Z}_p} (1+t)^{a_1x_1} d\mu_0(x_1) = \frac{a_1t}{(1+t)^{a_1nt}-1} \sum_{l=0}^{n-1} (1+t)^{a_1l}. \tag{2.7}$$

From (2.7), we can derive

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{(a_1x_1+\cdots+a_rx_r)}d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&=\left(\frac{a_1t}{(1+t)^{na_1t}-1}\right)\cdots\left(\frac{a_rt}{(1+t)^{na_rt}-1}\right)\sum_{l_1,\cdots,l_r=0}^{n-1}(1+t)^{(a_1l_1+\cdots+a_rl_r)}\\
&=\left(\prod_{i=1}^{r}a_i\right)\sum_{l_1,\cdots,l_r=0}^{n-1}\left(\sum_{k=0}^{\infty}D_k(na_1,\cdots,na_r)\frac{t^k}{k!}\right)\sum_{j=0}^{\infty}(a_1l_1+\cdots+a_rl_r)^j\frac{t^j}{j!}\\
&=\left(\prod_{i=1}^{r}a_i\right)\sum_{m=0}^{\infty}\left\{\sum_{l_1,\cdots,l_r=0}^{n-1}\sum_{j=0}^{m}(a_1l_1+\cdots+a_rl_r)^jD_{m-j}(na_1,\cdots,na_r)\binom{m}{j}\right\}\frac{t^m}{m!}.
\end{aligned}\tag{2.8}$$

By (2.8), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a_1x_1+\cdots+a_rx_r)^m d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&=(\prod_{i=1}^{r}a_i)\sum_{l_1,\cdots,l_r=0}^{n-1}\sum_{j=0}^{m}(a_1l_1+\cdots+a_rl_r)^jD_{m-j}(na_1,\cdots,na_r)\binom{m}{j},
\end{aligned}\tag{2.9}$$

where $n\in\mathbb{N}$ and $m\in\mathbb{Z}\geq 0$.

Therefore, by Theorem 2.1 and (2.9), we obtain the following theorem.

Theorem 2.4. For $n\in\mathbb{N}$ and $m\in\mathbb{Z}$ with $m\geq 0$, we have

$$D_m(a_1,\cdots,a_r)=\sum_{l_1,\cdots,l_r=0}^{n-1}\sum_{j=0}^{m}(a_1l_1+\cdots+a_rl_r)^jD_{m-j}(na_1,\cdots,na_r)\binom{m}{j}.$$

Moreover,

$$D_m(x|a_1,\cdots,a_r)=\sum_{l_1,\cdots,l_r=0}^{n-1}\sum_{j=0}^{m}(a_1l_1+\cdots+a_rl_r+x)^jD_{m-j}(na_1,\cdots,na_r)\binom{m}{j}.$$

From (2.8), we observe that

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{(a_1x_1+\cdots+a_rx_r)}d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&=\left(\frac{a_1t}{(1+t)^{na_1}-1}\right)\cdots\left(\frac{a_rt}{(1+t)^{na_r}-1}\right)\sum_{l_1,\cdots,l_r=0}^{n-1}(1+t)^{(a_1l_1+\cdots+a_rl_r)}\\
&=\sum_{l_1,\cdots,l_r=0}^{n-1}\frac{a_1t\cdots a_rt}{((1+t)^{na_1}-1)\cdots((1+t)^{na_r}-1)}(1+t)^{(\frac{a_1l_1+\cdots+a_rl_r}{n})n}\\
&=\left(\prod_{i=1}^{r}a_i\right)\sum_{l_1,\cdots,l_r=0}^{n-1}\sum_{m=0}^{\infty}D_m\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\Big|a_1,\cdots,a_r\right)n^m\frac{t^m}{m!}\\
&=\sum_{m=0}^{\infty}\left(\prod_{i=1}^{r}a_i\right)n^m\sum_{l_1,\cdots,l_r=0}^{n-1}D_m\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\Big|a_1,\cdots,a_r\right)\frac{t^m}{m!}.
\end{aligned}\tag{2.10}$$

Thus, by (2.10), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a_1x_1+\cdots+a_rx_r)^m d\mu_0(x_1)\cdots d\mu_0(x_r)\\
&=\left(\prod_{i=1}^{r}a_i\right)n^m\sum_{l_1,\cdots,l_r=0}^{n-1}D_m\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\Big|a_1,\cdots,a_r\right),
\end{aligned}\tag{2.11}$$

where $n\in\mathbb{N}$ and $m\in\mathbb{Z}\geq 0$.

Therefore, by Theorem 2.1 and (2.10), we obtain the following theorem.

Theorem 2.5. For $n\in\mathbb{N}$ and $m\geq 0$, we have

$$D_m(a_1,\cdots,a_r)=n^m\sum_{l_1,\cdots,l_r=0}^{n-1}B_m\left(\frac{a_1l_1+\cdots+a_rl_r}{n}\Big|a_1,\cdots,a_r\right).$$

Moreover,

$$D_m(x|a_1,\cdots,a_r)=n^m\sum_{l_1,\cdots,l_r=0}^{n-1}D_m\left(\frac{a_1l_1+\cdots+a_rl_r+x}{n}\Big|a_1,\cdots,a_r\right).$$

References

[1] A. Bayad, T.Kim, *Results on values of Barnes polynomials*, Rocky Mt. J. Math. 43(6)(2013) 1857–1869.

[2] Y.S. Jang, H.I. Kwon, S.-H. Rim, J.J. Seo, *A note on q-analogue of lambda-Daehee polynomials*, Adv. Studies. Theor. Phys.**8** (2014) no.13, 589–597.

[3] D. S. Kim, T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci.(Ruse). 7(2013), no. 120, 5969–5976.

[4] D. S. Kim, T. Kim, *Identities arising from higher-order Daehee polynomial bases*,Open Math.**13**(2015), Art. 19.

[5] D. S. Kim, T. Kim, H.I. Kwon, J. J. Seo, *Daehee Polynomials with q-parameter*, Adv.Studies Theor. Phys., **8** (2014), no.13, 561–569.

[6] D.S. Kim, T. Kim, S.-H. Lee, J. J. Seo, *Higher-order Daehee numbers and polynomials*, Int. J. Math. Anal.(Ruse) **8** (2014), no. 6, 273–283.

[7] D. S. Kim, T. Kim, J. J. Seo, *Higher order Daehee polynomials of the first kind with umbral calculus*, Adv. Stud. Contemp. Math. 24(1)(2014) 5–18.

[8] T. Kim, *On Euler-Barnes multiple zeta functions*, Russ. J. Math. Theor. 10(3)(2003) 261–267.

[9] T. Kim, *Barnes-type multiple q-zeta functions and q-Euler polynomials*, J. Phys. A: Math.Theor. 43(25)(2010) 11p.

[10] T. Kim, S.H. Lee, T. Mansour, J. J. Seo, *A note on q-Daehee polynomials and numbers*, Adv. Stud. Contemp. Math. **24**(2014), no.2, 155–160.

[11] D. Lim, Y. Do, *Some identities of Barnes-type special polynomials*, Adv. Difference Equ. 2015:42,(2015).

[12] D. Lim, Y. Do, *On (h,q)-Daehee numbers and polynomials*, Adv. Difference Equ. 2015:107,(2015).

[13] J.-W. Park, *On the twisted Daehee polynomials with q-parameter*, Adv. Difference Equ. 2014:304,(2014).

[14] J.-W. Park, S.-H. Rim, J. Kwon, *The twisted Daehee numbers and polynomials*, Adv. Difference Equ. 2014:1,(2014).

[15] C.S.Ryoo, *In the Barnes' type multiple Euler polynomials*, Proc. Jangjeon Math.Soc. 13(3)(2010) 327–335.