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Common Fixed Point and Best Approximation Results for Generalized Asymptotically (f, g) -Nonexpansive Mappings on Starshaped versus Nonstarshaped Domains

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Abstract:

In the present paper we obtain a common fixed point theorem for generalized asymptotically (f, g)- nonexpansive mappings in the convex metric space and apply it to find new best approximation results for such type of mappings on starshaped as well as nonstarshaped domains. The results proved in this paper unify, generalize and complement various known existing results in the literature to a more general class of noncommuting mappings.

Keywords: Convex metric space, Common fixed point, C_q-commuting mapping, generalized asymptotically (f, g) - nonexpansive mapping, contractive jointly continuous family and Best approximation.

Mathematics Subject Classification: 54H25, 41A50, 47H10.

1. INTRODUCTION

Common fixed points of two commuting mappings satisfying some contractive or nonexpansive type conditions have been studied by many researchers (see [1]-[4], [11], [14]-[16] and references cited therein). The introduction of noncommuting mappings such as weakly commuting, R-weakly commuting, R-subweakly commuting, compatible, weakly compatible and C_q-commuting mappings was a turning point in the fixed point arena. A wider class of nonexpansive mapings known as asymptotically nonexpansive mappings was introduced by Goebel and Kirk [8]. Vijayaraju and Hemavathy [24] proved some common fixed point theorems and approximation results by extending the results of Beg et al. [2] to generalized asymptotically S-nonexpansive and C_q-commuting mappings in normed linear spaces. Chandok and Narang [5] generalised the result of Vijayaraju and Hemavathy [24] to convex metric spaces.

In this context, it may also be mentioned that Dotson [6] proved the existence of fixed point for nonexpansive mapping in the setup of starshaped. He further extended his result without starshapedness under non-convex condition [7]. This idea was utilized by Mukherjee and Som [13] to prove existence of fixed point as best approximant. In this way, they extended the result of Singh [21] without starshapedness condition.

The purpose of this paper is to prove common fixed point theorems for generalized asymptotically (f, g)-nonexpansive and C_q-commuting mappings in the setting of convex metric spaces on starshaped as well as nonstarshaped domains. As an application, the results on the set of best approximation are also obtained which extend, improve and generalize the results of Hussain [11], Dotson [7], Habiniak [10], Imdad [12], Nashine and Shrivastva [17], Sahab et. al.[18], Singh [21, 22] etc.

2. DEFINITIONS AND PRELIMINARIES

Definition 2.1. Let (X, d) be a metric space. A continuous mapping $W: X \times X \times [0, 1] \to X$ is said to be a convex structure on X, if for all x, y ∈ X and $\lambda \in [0, 1]$, the following condition is satisfied:

$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, y)$ for all u ∈ X.

A metric space X together with this convex structure is called a convex metric space [23].

Examples 2.2. Banach space and each of its convex subsets are simple examples of convex metric spaces with $W(x, y, \lambda) = \lambda x + (1-\lambda)y$.

Example 2.3. Let $X = \{(x_1, x_2) \in R^2: x_1, x_2 > 0\}$. For $x = (x_1, x_2)$, $y = (y_1, y_2)$ in X and $\alpha \in [0, 1]$ define a mapping $W: X \times X \times [0, 1] \to X$ by

$W(x, y, \alpha) = \left(\alpha x_1 + (1-\alpha)y_1, \frac{\alpha x_1 x_2 + (1-\alpha)y_1 y_2}{\alpha x_1 + (1-\alpha)y_1}\right)$ and a metric $d: X \times X \to [0, \infty)$ by

$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|$. It can be verified that X is a convex metric space but not a normed linear space.

Definition 2.4 [23]. A subset M of a convex metric space X is said to be **convex**, if $W(x, y, \lambda) \in M$ for all x, y ∈ M and $\lambda \in [0, 1]$.

Definition 2.5 [9]. The subset M of a convex metric space (X, d) is said to **q-starshaped** if there exists q ∈ M such that $W(q, x, \lambda) \in M$ for all x ∈ M and $\lambda \in [0, 1]$. In other words, the set M is called q- starshaped with q ∈ M if the segment $[q, x] = \{W(q, x, \lambda): 0 \leq \lambda \leq 1\}$ joining q to x, is contained in M for all x ∈ M.

Remark 2.6. Clearly, q-starshaped subsets of X contain all convex subsets of X as a proper subclass.

Definition 2.7. A convex metric space X is said to satisfy property (I) [9] if for all x, y ∈ X and λ ∈ [0, 1], we have d (W(p, x, λ),W(p, y, λ)) ≤ λ d(x, y), where p is arbitrary but fixed point of X.
Property (I) is always satisfied in normed linear spaces.

Definition 2.8. A continuous function T from a closed convex subset M of a convex metric space X into itself is said to be **affine** on M, if T(W(x, y, λ)) = W(Tx, Ty, λ) for all x, y ∈ M and λ ∈ (0, 1) ∩ Q, where Q denotes the set of rational numbers.

Definition 2.9. Let (X, d) be a metric space and S, T: X → X. A point x ∈ X is called:
(1) **a fixed point** of T if Tx = x,
(2) **a coincidence point** of the pair (S, T) if Tx = Sx,
(3) **a common fixed point** of the pair (S, T) if x = Tx = Sx.

We denote by **F(T) and C(S, T),** the set of all fixed points of T and the set of all coincidence points of the pair {S, T} respectively.

Definition 2.10. Let (X, d) be a metric space and S, T: X → X be two mappings. The mapping S and T are said to be weakly compatible if they commute their coincident point, i.e., TSx = STx whenever Sx = Tx.

Definition 2.11. Let (X, d) be a convex metric space, M be a q-starshaped subset of X and T, f and g be self mappings on X and q ∈ F(f), then T is said to be:
(1) **Uniformly asymptotically regular** on M if for each ε > 0 there exists a positive integer N such that $d(T^n x, T^n y) < \varepsilon$ for all n ≥ N and for all x in M.
(2) **R-subweakly commuting [19]** if there exists a real number R > 0 such that $d(Tfx, fTx) \leq Rd(fx, Y_q^{T(x)})$ where
$Y_q^{T(x)} = \{ y_\lambda : y_\lambda = W(Tx, q, \lambda)$ for all $x \in M\}$.
(3) **C_q-commuting [1]** if fTx = Tfx for all x ∈ C_q (f, T), where
$C_q(f, T) = \cup\{C(f, T_k): 0 \leq k \leq 1\}$ and $T_k(x) = W(Tx, q, k)$.
(4) **Asymptotically f-nonexpansive [8]** if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $d(T^n x, T^n y) \leq k_n\, d(fx, fy)$ for each x, y in M and each n ∈ ℕ. If f = I (identity map), then T is asymptotically nonexpansive mapping.
(5) **Generalized asymptotically f-nonexpansive [24]** if there exists a sequence $\{k_n\}$ of real numbers in $[1,\infty)$ with $k_n \geq k_{n+1}$, $k_n \to 1$ as $n\to\infty$ such that
$$d(T^n x, T^n y) \leq k_n \max\{d(fx, fy), dist(fx, [T^n x, q]), dist(fy, [T^n y, q]), \tfrac{1}{2}[dist(fx, [T^n y, q])+dist(fy, [T^n x, q])]\}$$
for all x, y ∈ M and n ∈ ℕ.

(6) **Generalized asymptotically (f, g)-nonexpansive** if there exists a sequence $\{k_n\}$ of real numbers in $[1,\infty)$ with $k_n \geq k_{n+1}$, $k_n \to 1$ as $n \to \infty$ such that
$$d(T^n x, T^n y) \leq k_n \max\{d(fx, gy), \text{dist}(fx, [T^n x, q]), \text{dist}(gy, [T^n y, q]), \frac{1}{2}[\text{dist}(fx, [T^n y, q]) + \text{dist}(gy, [T^n x, q])]\}$$
for all x, y ∈ M and $n \in \mathbb{N}$. If f = g then T is generalized asymptotically f-nonexpansive mapping.

Remark 2.12. Every $C_q -$ commuting mapping is weakly compatible but converse need not be true. For example, let X = $\mathbb{R}$ with usual norm and M = [1, ∞). Let f(x) = 2x − 1 and g(x) = x^2 for all x ∈ M. Let q = 1. Then M is q-starshaped with fq = q and C_q(f, g) = [1, ∞). Note that f and g are weakly compatible maps but f and g are not C_q-commuting mappings.

(ii) Clearly, R-subweakly commuting mappings are C_q-commuting but converse does not hold. For example, let X = R be endowed with the usual metric and M = [0, ∞). Define T, f: M → M by

$$fx = \begin{cases} \frac{x}{2} & 0 \leq x < 1, \\ x & x \geq 1 \end{cases}$$

and

$$Tx = \begin{cases} \frac{1}{2} & 0 \leq x < 1, \\ x^2 & x \geq 1 \end{cases}$$

Then M is q-starshaped with q = 1 and C_q (T, f) = [1, ∞). Moreover f and T are C_q-commuting but not R-subweakly commuting.

Remark 2.13 [15]. If T and f are C_q-commuting on M, then $fT^n x = T^n fx$ for all $x \in C_q(f, T^n)$, where $C_q(f, T^n) = \cup\{C(f, T_{k_n}): 0 \leq k_n \leq 1\}$ and $T_{k_n}x = \{W(Tx, q, k_n): 0 \leq k_n \leq 1\}$.

Dotson [7] proved some results concerning the existence of fixed point of nonexpansive mappings on a certain class of nonconvex sets. For proving these results which extended his previous work [6] on starshaped sets, he introduced the following class of nonconvex sets:

Definition 2.14. Let M be a subset of a Banach space and $\{f_\alpha: \alpha \in M\}$ be family of maps from [0, 1] into M having property that for each α ∈ M we have $f_\alpha(1) = \alpha$. Such a family is said to be:

1. **contractive** provided there exist a function φ: (0, 1) → (0, 1) such that for all α, β ∈ Mand for all t ∈ (0, 1), we have $d(f_\alpha(t), f_\beta(t)) \leq \varphi(t) d(\alpha, \beta)$;
2. **jointly continuous** if $t \to t_0$ in [0, 1] and $\alpha \to \alpha_0$ in M imply $f_\alpha(t) \to f_{\alpha_0}(t_0)$.

Remark 2.15. This notion can easily be extended to metric spaces. Also it is easy to observe that if M is a starshaped (with z as star center) subset of a normed linear space X and $f_z(t) = tx + (1-t)z$, x ∈ M, t ∈ [0, 1], then $\mathfrak{F} = \{f_\alpha\}_{\alpha \in M}$ is a contractive jointly continuous family with φ(t) = t. Thus the class of subsets of X with the property of contractive and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets.

Definition 2.16. A self mapping T of M is said to satisfy property (Γ), if the condition $T(f_\alpha(t)) = f_{T\alpha}(t)$ holds for all $f_\alpha \in \mathfrak{F}$, $\alpha \in M$ and $t \in [0, 1]$.

Definition 2.17. Let M be a closed subset of a metric space (X, d) and $x \in X$. If there exists an element y_0 in M such that $d(x, y_0) = d(x, M)$, then y_0 is called best approximation to x out of M. We denote by $P_M(x)$, the set of all best approximation to x out of M.

3. MAIN RESULTS

3.1 Common fixed point and best approximation results for generalized asymptotically (f, g)-nonexpansive mappings on starshaped domain:

In this section, we discuss the existence of common fixed point of generalized asymptotically (f, g)-nonexpansive mappings on the starshaped subset of a convex metric space. As an application, the results on the best approximation are also derived. The following result which is the consequence of the lemma 3.2 proved in [26] is needed in the sequel:

Lemma 3.1.1. Let M be a nonempty closed subset of a metric space (X, d). Let f, g, T be continuous self mappings on M, $q \in F(f) \cap F(g)$ and $T(M\backslash\{q\}) \subset f(M) \cap g(M)\backslash\{q\}$. Suppose there exists $k \in (0, 1)$ such that
$d(Tx, Ty) \leq k \max\{(fx, gy), d(fx, Tx), d(gy, Ty), \frac{1}{2}[d(fx, Ty) + d(gy, Tx)]\}$
for all $x, y \in M$. Further, if $cl[T(M\backslash\{q\})]$ is complete, the pairs {T, f} and {T, g} are weakly compatible on $M\backslash\{q\}$, then $F(f) \cap F(T) \cap F(g)$ is singleton.

Theorem 3.1.2. Let M be a nonempty q-starshaped subset of a convex metric space (X, d) with Property (I) and let f, g, T: M → M be continuous self mappings with $q \in F(f) \cap F(g)$ and f, g are affine with respect to q. If the pairs {T, f} and {T, g} are C_q-commuting on $M\backslash\{q\}$, $cl[T(M\backslash\{q\})] \subseteq f(M) \cap g(M)\backslash\{q\}$, $cl[T(M\backslash\{q\})]$ is compact and T is uniformly asymptotically regular and generalized asymptotically (f, g)-nonexpansive map with sequence $\{k_n\}$, then $F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. Let $\{\lambda_n\}$ be a sequence in [0, 1) with $\lim_{n\to\infty} \lambda_n = 1$. Take $\alpha_n = \frac{\lambda_n}{k_n}$, then $\{\alpha_n\}$ is a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\alpha_n \to 1$. Define T_n as $T_n x = W(T^n x, q, \alpha_n)$, for all $x \in M$ and for each $n \geq 1$. As M is q-starshaped, f and g are affine with respect to q and $cl[T(M\backslash\{q\})] \subseteq f(M) \cap g(M)\backslash\{q\}$, therefore T_n is a self mapping of M and $cl[T_n(M\backslash\{q\})] \subseteq f(M) \cap g(M)\backslash\{q\}$ for each n.

Since the pair {T, f} is C_q-commuting, $fT^n x = T^n f x$ for all $x \in C_q(f, T^n)$ and f is affine with respect to q, therefore for each $x \in C(f, T_n) \subseteq C_q(f, T)$,
$fT_n x = f(W[T^n x, q, \alpha_n]) = W[fT^n x, fq, \alpha_n] = W[T^n fx, q, \alpha_n] = T_n f x$.

Hence the pair $\{f, T_n\}$ is weakly compatible for all n. Similarly the pair $\{g,T_n\}$ is weakly compatible for all n. Further, we have

$$\begin{aligned} d(T_n x, T_n y) &= d(W(T^n x, q, \alpha_n), W(T^n y, q, \alpha_n)) \\ &\le \alpha_n d(T^n x, T^n y) \\ &\le \alpha_n k_n \max\{d(fx, gy), \text{dist}(fx, [T^n x, q]), \text{dist}(gy, [T^n y, q]), \\ &\qquad \tfrac{1}{2}[\text{dist}(fx, [T^n y, q]) + \text{dist}(gy, [T^n x, q])]\} \\ &\le \lambda_n \max\{d(fx, gy), d(fx, T_n x), d(gy, T_n y), \tfrac{1}{2}[d(fx, T_n y) + d(gy, T_n x)]\} \end{aligned}$$

for all $x, y \in M$.

As $cl[T(M\backslash\{q\})]$ is compact, each $cl[T_n(M\backslash\{q\})]$ is also compact. By Lemma 3.1.1, there exists $x_n \in M$ such that $fx_n = gx_n = T_n x_n = x_n$. Since $\{T_n x_n\}$ is a sequence in the compact set $cl[T_n(M\backslash\{q\})]$, there exists a subsequence $\{T_{n_i} x_{n_i}\}$ of $\{T_n x_n\}$ such that $\{T_{n_i} x_{n_i}\} \to z$, for some $z \in cl[T(M\backslash\{q\})]$. Moreover,

$x_{n_i} = f x_{n_i} = g x_{n_i} = T_{n_i} x_{n_i} = W[T^{n_i} x_{n_i}, q, \alpha_{n_i}] \to z$.

Since f and g are continuous, $x_{n_i} = fx_{n_i} \to fz$ and $x_{n_i} = gx_{n_i} \to gz$. By the uniqueness of the limit $z = fz = gz$. As T is continuous, $T^{n_i} x_{n_i} \to T^{n_i} z$. Again by the uniqueness of the limit, we have $\lim T^{n_i} z = z$ and $\lim T^{n_i+1} = Tz$. Hence it follows that $d(z, Tz) \le d(z, T^{n_i+1} z) + d(T^{n_i} z, T^{n_i+1} z) + d(T^{n_i+1} z, Tz)$ which tends to zero as $n \to \infty$. Therefore, $Tz = z = f z = gz$. Hence, $F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Corollary 3.1.3.[5] Let M be a nonempty subset of a convex metric space (X, d) with Property (I) and let f, T: $M \to M$ be continuous self mappings with $cl[T(M\backslash\{q\})) \subseteq f(M) \backslash\{q\}$. Suppose that M is q-starshaped with $q \in F(f)$ and f is affine with respect to q. If the pair {T, f} is C_q-commuting on $M\backslash\{q\}$, $cl[T(M\backslash\{q\})]$ is compact, and T is uniformly asymptotically regular and generalized asymptotically f-nonexpansive mapping with sequence $\{k_n\}$, then $F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. By taking f = g, the result follows.

Remark 3.1.4. The above theorem also generalises the results of Shahzad [19], Narang and Chandok [15] and Al-Thagafi and Shahzad [1].

We now give an application of Theorem 3.1.2 to the set of best approximation.

Theorem 3.1.5. Let M be a nonempty subset of a convex metric space (X, d) with Property (I) and T, f and g be continuous self mappings of X such that $T(\partial M \cap M) \subseteq M$ and u be common fixed point of f, g and T for some $u \in X\backslash M$, where ∂M denotes boundary of M. Suppose that $P_M(u)$ is closed, q-starshaped, f and g are affine with respect to $q \in F(f) \cap F(g)$ and $f(P_M(u)) = g(P_M(u)) = P_M(u)$. If the pairs {T, f} and {T, g} are C_q-commuting on $P_M(u) \cup \{u\}$ with $d(Tx, Tu) \le d(fx, gu)$, $cl[T(P_M(u))]$ is compact and T is uniformly asymptotically regular and generalized asymptotically (f, g)-nonexpansive mapping for all $x, y \in P_M(u)$. Then $P_M(u) \cap F(T) \cap F(f) \cap F(g) \neq \emptyset$.

Proof. Let $x \in P_M(u)$, then for any $k \in (0, 1]$, we have $d(W(u, x, k), u) \le kd(u, u) + (1-k)\, d(x, u) = (1-k)d(x, u) < dist(u, M)$.
This implies that $W(u, x, k) \notin M$ for any $k \in (0, 1]$. It follows that the open line segment $\{W(u, x, \lambda): 0 < \lambda < 1\}$ and the set M are disjoint. Thus x is not in the interior of M and so $x \in \partial M \cap M$.
Since $T(\partial M \cap M) \subseteq M$, Tx must be in M. Also, $f(P_M(u)) = P_M(u)$, $fx \in P_M(u)$ and u is common fixed point of f, g and T, therefore from the given contractive condition, we obtain
$d(Tx, u) = d(Tx, Tu) \le d(fx, gu) = d(fx, u) = d(u, M)$.
Thus $P_M(u)$ is T-invariant. Hence, $T(P_M(u)) \subset P_M(u) = f(P_M(u)) = g(P_M(u))$ and therefore, by theorem 3.1.2 there exists $z \in P_M(u)$ such that $P_M(u) \cap F(T) \cap F(f) \cap F(g) \ne \emptyset$.

Theorem 3.1.6. Let M be a nonempty subset of a convex metric space (X, d) with Property (I) and T, f and g be continuous self mappings of X such that $T(\partial M \cap M) \subseteq M$ and u be common fixed point of f, g and T for some $u \in X\backslash M$, where ∂M denotes boundary of M. Suppose that f and g are affine with respect to $q \in F(f) \cap F(g)$ and $f(D) = g(D) = D$, where $D = P_M(u) \cap C_M^f(u) \cap C_M^g(u)$ $[C_M^f(u) = \{x \in M: fx \in P_M(u)\}$ and $C_M^g(u) = \{x \in M: gx \in P_M(u)\}]$ is closed and q-starshaped,. If the pairs {T, f} and {T, g} are C_q-commuting on $P_M(u) \cup \{u\}$ with $cl[T(P_M(u))]$ is compact, $d(Tx, Tu) \le d(fx, gu)$, f and g are nonexpansive on $P_M(u) \cup \{u\}$, T is uniformly asymptotically regular and generalized asymptotically (f, g)- nonexpansive mapping for all $x, y \in D \cup \{u\}$, then $D \cap F(T) \cap F(f) \cap F(g) \ne \emptyset$.

Proof: Let $x \in D$, then proceeding as in Theorem 3.1.5, we shall get $Tx \in P_M(u)$. As f and g are nonexpansive on $D \cup \{u\}$, we obtain
$d(fTx, u) = d(fTx, fu) \le d(Tx, u) < d(u, M)$.
Thus, $fTx \in P_M(u)$. This implies that $Tx \in C_M^f(u)$. Similarly we can show that $Tx \in C_M^g(u)$ and hence $Tx \in D$ i.e., D is T-invariant. Since all the conditions of Theorem 3.1.2 are satisfied, therefore, $D \cap F(T) \cap F(f) \cap F(g) \ne \emptyset$.

Remark 3.1.7. Theorems 3.1.5 and 3.1.6 extend the corresponding results of [14]-[16], [19] and [25] to generalized asymptotically (f, g)-nonexpansive mappings.

3.2 Common fixed point and best approximation results for generalized asymptotically (f, g)-nonexpansive mappings on nonstarshaped domain:

In this section, we discuss the existence of common fixed point from the set of best approximation for generalized asymptotically (f, g)-nonexpansive mappings on a subset (not necessarily starshaped) having contractive jointly continuous family introduced by Dotson [7].

Theorem 3.2.1. Let T, g and h be self mappings of a convex metric space X and M be a subset of X such that $T(\partial M \cap M) \subseteq M$. Let $u \in F(T) \cap F(g) \cap F(h)$. Further, suppose that $D = P_M(u)$ is nonempty, compact and has a contractive jointly continuous family $\mathfrak{I} = \{f_x : x \in D\}$ such that g and h satisfy property (Γ) for all $x \in D$ and $t \in [0, 1]$, T is generalized asymptotically (g, h) - nonexpansive map with sequence $\{k_n\}$ for all $x, y \in D \cup \{u\}$ and $g(D) = D = h(D)$. If the pair (T, g) and (T, h) are weakly compatible with $d(Tx, Tu) \leq d(gx, hu)$ and any one of T, g and h is continuous, then $D \cap F(T) \cap F(g) \cap F(h) \neq \emptyset$.

Proof. First, we show that T is a self map on D. For this, let $x \in D$. Then proceeding as in Theorem 3.1.5, we shall get $Tx \in P_M(u)$. Now, for each $n \geq 1$, define $T_n : M \to M$ by $T_n(x) = f_{T^n x}(k_n)$, for all $x \in D$, where $\{k_n\}$ is a sequence in (0, 1) such that $k_n \to 1$. Then $\{T_n\}$ is a well-defined map from D into D for each n. Since the mapping g satisfies the property (Γ), we have

$$T_n g(x) = f_{T^n gx}(k_n) = f_{gT^n x}(k_n) = g f_{T^n x}(k_n) = gT_n(x).$$

Hence $\{T_n\}$ and g are weakly compatible on D for each n and $T_n(D) \subseteq D = g(D)$. Similarly it can be shown that $\{T_n\}$ and h are weakly compatible on D for each n and $T_n(D) \subseteq D = h(D)$.

Now, the definition of T_n and contractiveness of $\mathfrak{I}$ imply that

$$\begin{aligned} d(T_n x, T_n y) &= d(f_{T^n x}(k_n), f_{T^n y}(k_n)) \\ &\leq \varphi(k_n) d(T^n x, T^n y) \\ &< d(T^n x, T^n y) \\ &\leq k_n \max\{d(gx, hy), \mathrm{dist}(gx, [T^n x, q]), \mathrm{dist}(hy, [T^n y, q]), \\ &\qquad \tfrac{1}{2}[\mathrm{dist}(gx, [T^n y, q]) + \mathrm{dist}(hy, [T^n x, q])]\} \\ &\leq k_n \max\{d(gx, hy), d(gx, T_n x), d(hy, T_n y), \tfrac{1}{2}[d(gx, T_n y) + d(hy, T_n x)]\} \end{aligned}$$

for all $x, y \in D$.

As D is compact, therefore by lemma 3.1.1, $F(T_n) \cap F(g) \cap F(h) = \{x_n\}$ for each n. Also, since D is compact, there exists a subsequence of $\{x_n\}$ in D, denoted by $\{x_m\}$, converging to a point, say, $y \in D$ and hence $T^m x_m \to T^m y$. The jointly continuity of $\mathfrak{I}$ gives $x_m = T_m x_m = f_{T^m x_m}(k_m) \to f_{T^m y}(1) = T^m y$ and thus the uniqueness of the limit implies $T^m y = y$ and $T^{m+1} y = Ty$. Hence it follows that $d(y, Ty) \leq d(y, T^m y) + d(T^m y, T^{m+1} y) + d(T^{m+1} y, Ty) \to 0$, giving thereby, $y \in D \cap F(T)$, provided T is taken to be continuous. Now since $T(D) \subseteq g(D)$ and $T(D) \subseteq h(D)$, there exists a point z in X such that $Ty = y = gz$.

For $x_0 \in D$ arbitrary, there exists $x_1 \in D$ be such that $Tx_0 = gx_1$ and for this point x_0, there exists a point x_2 in D such that $Tx_1 = hx_2$ and so on. Inductively, one can choose x_n such that $Tx_{2n} = gx_{2n+1}$ and $Tx_{2n+1} = hx_{2n+2}$. Therefore, we get

$$\begin{aligned} d(Tz, Tx_{2n+2}) \leq k_n \max\{&d(gz, hx_{2n+2}), \mathrm{dist}(gz, [T^1 z, q]), \mathrm{dist}(hx_{2n+2}, [T^1 x_{2n+2}, q]), \\ &\tfrac{1}{2}[\mathrm{dist}(gz, [T^1 x_{2n+2}, q]) + \mathrm{dist}(hx_{2n+2}, [T^1 z, q])]\} \end{aligned}$$

which on letting $n \to \infty$, one gets $Tz = gz = y$.

Since (T, g) are weakly compatible, therefore $d(Ty, gy) = d(Tgz, gTz) = 0$, yielding thereby $Ty = gy$. Hence $gy = Ty = y$. Also since $T(D) \subseteq h(D)$, there exists a point v in X and $Ty = y = hv$. Now,

$$\begin{aligned} d(Tx_{2m+1}, Tv) \leq k_n \max\{&d(gx_{2m+1}, hv), \mathrm{dist}(gx_{2m+1}, [T^1 x_{2m+1}, q]), \mathrm{dist}(hv, [T^1 v, q]), \\ &\tfrac{1}{2}[\mathrm{dist}(gx_{2m+1}, [T^1 v, q]) + \mathrm{dist}(hv, [T^1 x_{2m+1}, q])]\} \end{aligned}$$

which on letting $m \to \infty$, reduced to $y = Tv = hv$. Since (T, h) are weakly compatible, therefore $hy = h(Tv) = T(hv) = y$, which show that $y \in D$ is a common fixed point of T, g and h. Hence $D \cap F(T) \cap F(g) \cap F(h) \neq \emptyset$. This completes the proof.

Theorem 3.2.2. Let M be a nonempty subset of a convex metric space (X, d) and T, g and h be continuous self mappings of X such that $T(\partial M \cap M) \subseteq M$ and u be common fixed point of g, h and T for some $u \in X\backslash M$. Further, suppose that g and h satisfy property (Γ) for all $x \in D$, $t \in [0, 1]$ and $g(D) = D = h(D)$, where $D = P_M(u) \cap C_M^g(u) \cap C_M^h(u)$ $[C_M^g(u) = \{x \in M: gx \in P_M(u)\}$ and $C_M^h(u) = \{x \in M: hx \in P_M(u)\}]$ is nonempty compact and has a contractive jointly continuous family $\Im = \{f_x: x \in M\}$. If the pair (T, g) and (T, h) are weakly compatible with $d(Tx, Tu) \leq d(gx, hu)$, g and h are nonexpansive on $P_M(u) \cup \{u\}$, T is generalized asymptotically (g, h)-nonexpansive mapping with sequence $\{k_n\}$ for all $x, y \in D \cup \{u\}$and any one of T, g and h is continuous, then $D \cap F(T) \cap F(g) \cap F(h) \neq \emptyset$.

Proof: Let $x \in D$, then proceeding as in Theorem 3.1.5, we shall get $Tx \in P_M(u)$. As g and h are nonexpansive on $D \cup \{u\}$, we obtain
$d(gTx, u) = d(gTx, gu) \leq d(Tx, u) < d(u, M)$.
Thus, $gTx \in P_M(u)$. This implies that $Tx \in C_M^g(u)$. Similarly we can show that $Tx \in C_M^h(u)$ and hence $Tx \in D$ i.e., D is T-invariant. Since all the conditions of Theorem 3.2.1 are satisfied, therefore, $D \cap F(T) \cap F(g) \cap F(h) \neq \emptyset$.

Remark 3.2.3. We know that if M is a q-starshaped subset of a convex metric space (X, d) with Property (I) and $\Im$ be the family defined as $f_x(t) = W(x, p, t)$ for $0 < t < 1$, then this family is contractive jointly continuous by taking $\varphi(t) = t$. Moreover, if g and h are affine and $q \in F(g) \cap F(h)$, then $g(f_x(t)) = f_{gx}(t)$ and $h(f_x(t)) = f_{hx}(t)$, i.e., g and h satisfy property (Γ) for all $x \in M$ and $t \in [0, 1]$.

Thus, Theorems 3.2.1 and 3.2.2 generalize the Theorems 3.1.5 and 3.1.6 to nonstarshaped domain.

Remark 3.2.4. Theorems 3.2.1 and 3.2.2 also generalize the results of Hussain [11], Imdad [12], Nashine and Shrivastava [17] and Sahab, Khan and Sessa [18] by increasing the number of mappings and by employing the generalized form of nonexpansive mapping in a domain which is not necessarily starshaped and mappings are not necessarily linear.

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