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Suzuki Type Fixed Point Theorem For Rational Contraction In Partial Metric Spaces

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Abstract

In this paper, we obtain a Suzuki type unique common fixed point theorem by using rational contraction in partial metric spaces. Also we give an example which supports our theorem.

Key words: Partial metric, weakly compatible maps, Suzuki type contraction

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1. Introduction

The notion of a partial metric space was introduced by Matthews [9] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation and domain theory in computer science (see e.g. [5]).

Matthews [9] and Romaguera [11] and Altun et al. [2] proved some fixed point theorems in partial metric spaces for a single map.

For more works on fixed, common fixed point theorems in partial metric spaces, also we refer [1, 3, 4, 6, 7, 8, 10, 12, 13]).

The aim of this paper is to study Suzuki type unique common fixed point theorem of Jungck type maps by using rational contraction condition in partial metric spaces.

First we give the following theorem of Suzuki [13].

Theorem 1(See [13]): Let (X, d) be a complete metric space and let T be a mapping on X. Define a non - increasing function θ from [0, 1) onto $(\frac{1}{2},1]$ by

$$\theta(r)=\begin{cases} 1 & if \quad 0\le r\le \frac{\sqrt{5}-1}{2}, \\ \frac{r-1}{r^2} & if \quad \frac{\sqrt{5}-1}{2}\le r\le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & if \quad \frac{1}{\sqrt{2}}\le r<1. \end{cases}$$

Assume that there exists r ∈ [0, 1) such that
$\theta(r)d(x,Tx)\le d(x,y)$ implies $d(Tx,Ty)\le d(x,y)$ for all $x,y\in X$.
Then there exists a unique fixed point z of T.
Moreover $\lim_{n\to\infty} T^n x = z$ for all $x\in X$.
First we recall some basic dentitions and lemmas which play crucial role in the theory of partial metric spaces.

Definition 1.1. (See [9]) A partial metric on a nonempty set X is a function $p: X\times X \to R^+$ such that for all $x,y,z\in X$:

$(p_1) x=y \Leftrightarrow p(x,x)=p(x,y)=p(y,y),$
$(p_2) p(x,x)\le p(x,y), p(y,y)\le p(x,y),$
$(p_3) p(x,y)=p(y,x),$
$(p_4) p(x,y)\le p(x,z)+p(z,y)-p(z,z).$

The pair (X,P) is called a partial metric space (PMS).
If p is a partial metric on X, then the function $d_p : X\times X\to R^+$ given by

(1.1) $$d_p(x,y)=2p(x,y)-p(x,x)-p(y,y),$$

is a metric on X.

Example 1.2. (See e.g. [1, 7, 9]) Consider X = [0, ∞) with $p(x,y)=\max\{x,y\}$.Then (X, p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case
$d_p(x,y)=|x-y|$.

Example 1.3. (See [6]) Let $X=\{[a,b]: a,b\in R, a\le b\}$ and define $p([a,b],[c,d])=\max\{b,d\}-\max\{a,c\}$. Then (X, p) is a partial metric space.

We now state some basic topological notions (such as convergence, completeness, continuity) on partial metric spaces (see e.g. [1, 2, 7, 8, 9].)

Definition 1.4.

(i) A sequence $\{x_n\}$ in the PMS (X, p) converges to the limit x if and only if

$p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

(ii) A sequence $\{x_n\}$ in the PMS (X, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

(iii) A PMS (X, p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_m,x_n)$.

The following lemma is one of the basic results in PMS([1, 2, 7, 8, 9]).

Lemma 1.5.
(i) A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p).

(ii) A PMS (X, p) is complete if and only if the metric space (X, d_p) is complete.

Moreover $\lim_{n\to\infty} d_p(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_m,x_n)$.

Next, we give two simple lemmas which will be used in the proof of our main result. For the proofs we refer to [1].

Lemma 1.6. Assume $x_n \to z$ as n → ∞ in a PMS (X, p) such that $p(z,z) = 0$. Then $p(z,y) = \lim_{n\to\infty} p(x_n,y)$ for every $y \in X$.

Lemma 1.7. Let (X, p) be a PMS. Then

(A)If $p(x,y) = 0$ then $x = y$,

(B)If $x \neq y$ then $p(x,y) > 0$.

Remark 1.8. If $x = y$ then $p(x,y)$ may not be 0.

Definition 1.9. A pair (T, g) is called weakly compatible pair if they commute at coincidence points.

Now we prove our main result.

2. Main Result

Theorem 2.1. Let (X, p) be a partial metric space and let $T, g : X \to X$ be mappings satisfying

(2.1.1) if there exists a constant $\theta \in [0, 1)$ such that

$\eta(\theta) p(gx,Tx) \le p(gx,gy)$ implies that

$$p(Tx,Ty) \le \theta \max\left\{\frac{p(gx,Tx)p(gy,Ty)}{1+p(gx,gy)},\quad p(gx,gy)\right\},\ \forall x,y \in X,$$

where $\eta(\theta) : [0,1) \to (\frac{1}{2},1]$ defined by $\eta(\theta) = \frac{1}{1+\theta}$ is a strictly decreasing function,

(2.1.2) $T(X) \subseteq g(X)$ and either $T(X)$ or $g(X)$ is complete,

(2.1.3) the pair (T, g) is weakly compatible.

Then T and g have a unique common fixed point.

Proof. Let $x_0 \in X$ from (2.1.2) there exists a sequence $\{x_n\}$ in X such that $gx_{n+1} = Tx_n$ for all n = 0, 1, 2, 3, ...

Case(i): Assume that $gx_{n+1} \neq gx_n$, ∀ n.

Since $\eta(\theta) \leq 1$, we have

$\eta(\theta)p(gx_0,Tx_0) \leq p(gx_0,Tx_0) = p(gx_0,gx_1).$

From (2.1.1), we have

$$p(Tx_0,Tx_1) \leq \theta \max\left\{\frac{p(gx_0,Tx_0)p(gx_1,Tx_1)}{1+p(gx_0,Tx_0)}, \quad p(gx_0,gx_1)\right\}$$

$$\leq \theta \quad p(gx_0,gx_1).$$

$p(gx_1,gx_2) = p(Tx_0.Tx_1) \leq \theta\ p(gx_0,gx_1).$

Since $\eta(\theta)p(gx_1,Tx_1) \leq p(gx_1,Tx_1) = p(gx_1,gx_2).$

From(2.1.1), we have

$$p(Tx_1,Tx_2) \leq \theta \max\left\{\frac{p(gx_1,Tx_1)p(gx_2,Tx_2)}{1+p(gx_1,Tx_1)}, \quad p(gx_1,gx_2)\right\}$$

$$\leq \theta\ p(gx_1,gx_2).$$

Therefore

$$p(gx_2,gx_3) = p(Tx_1.Tx_2) \leq \theta\ p(gx_1,gx_2) \leq \theta^2 p(gx_0,gx_1).$$

Continuing in this way, we have $p(gx_n,gx_{n+1}) \leq \theta^n p(gx_0,gx_1).$

Clearly

(2.1) $\qquad p(gx_n,gx_{n+1}) \to 0 \ \ as \ \ n \to \infty.$

Now for $m > n$, consider

$$\begin{aligned} p(gx_m,gx_n) &\leq p(gx_n,gx_{n+1}) + p(gx_{n+1},gx_{n+2}) + \ldots + p(gx_{m-1},gx_m) \\ &\leq (\theta^n + \theta^{n+1} + \cdots + \theta^{m-1})\ p(gx_0,gx_1) \\ &\leq \frac{\theta^n}{1-\theta} p(gx_0,gx_1) \\ &\to 0 \text{ as } n \to \infty, m \to \infty. \end{aligned}$$

Hence $\{gx_n\}$ is a Cauchy sequence in g(X).

Suppose g(X) is complete, $\{gx_n\}$ converges to some z in g(X).Hence there exists u ∈ X such that z = gu.

That is

$$\lim_{n\to\infty} p(gx_n, z) = p(z,z).$$

From Lemma 1.5 ,we have

(2.2) $$p(z,z) = \lim_{n\to\infty} p(gx_n, z) = \lim_{n,m\to\infty} p(gx_n, gx_m) = 0.$$

Since $gx_{n+1} \neq gx_n$, for all n and $gx_n \to gu$, it follows that $gx_n \neq gu$, for infinitely many n.

Claim : $p(gu,Tx) \leq \theta\, p(gu, gx)$ for x ∈ X with $gx \neq gu$.

Let x ∈ X with $gx \neq gu$ then by definition, $p(gu, gx) > 0$.

Since $gx_n \to gu$ as n → ∞, there exists a positive integer n_0 such that

$p(gu, gx_n) \leq \frac{1}{3} p(gu, gx)$ for all $n \geq n_0$.

Now for $n \geq n_0$.

$$\begin{aligned}
\eta(\theta)\, p(gx_n, Tx_n) &\leq p(gx_n, Tx_n) = p(gx_n, gx_{n+1}). \\
&\leq p(gx_n, gu) + p(gu, gx_{n+1}) \\
&\leq \frac{1}{3} p(gu, gx) + \frac{1}{3} p(gu, gx) = \frac{2}{3} p(gu, gx) \\
&\leq p(gu, gx) - \frac{1}{3} p(gu, gx) \\
&\leq p(gu, gx) - p(gu, gx_n) \\
&\leq p(gx_n, gx).
\end{aligned}$$

Then we have $p(Tx_n, Tx) \leq \theta \max\left\{ \frac{p(gx_n, Tx_n)\, p(gx, Tx)}{1 + p(gx, gx_n)},\quad p(gx, gx_n) \right\}$.

Letting n → ∞ and using (2.2), we get

(2.3) $$p(gu,Tx) \leq \theta\, p(gu, gx).$$

Hence the claim.

Now consider,

$$\begin{aligned}
p(gx,Tx) &\leq p(gx, gu) + p(gu,Tx) \\
&\leq p(gx, gu) + \theta\, p(gx, gu), \text{ from (2.3)} \\
&\leq (1+\theta)\, p(gx, gu)
\end{aligned}$$

$$\frac{1}{1+\theta}\, p(gx,Tx) \leq p(gx, gu).$$

Thus $\eta(\theta)\, p(gx,Tx) \leq p(gx, gu)$.

Hence from (2.1.1), we have

(2.4) $$p(Tx,Tu) \leq \theta \max\left\{ \frac{p(gx,Tx)\, p(gu,Tu)}{1 + p(gx, gu)},\quad p(gx, gu) \right\}.$$

Now, take $x = x_n$ in (2.4), we get

$$p(Tx_n, Tu) \le \theta \max\left\{ \frac{p(gx_n, Tx_n)\, p(gu, Tu)}{1 + p(gu, gx_n)}, \quad p(gu, gx_n) \right\}.$$

Letting n $\rightarrow \infty$ and using (2.2), (2.1) and Lemma 1.6, we get

$$p(gu, Tu) \le 0.$$

Therefore $gu = Tu = z$.

Since the pair (T, g) is weakly compatible, we have $gz = Tz$.

Suppose $Tz \ne z$.

$\eta(\theta)\, p(gu, Tu) = \eta(\theta)\, p(z, z) = 0 \le p(gz, gu)$.

Hence from (2.1.1), we have

$$p(Tz, Tu) \le \theta \max\left\{ \frac{p(gz, Tz)\, p(z, z)}{1 + p(gz, gu)}, \quad p(gu, gz) \right\}$$

$$p(Tz, z) \le \theta\ p(Tz, z) \quad \text{,from}(p_2)$$

which in turn yields that $gz = Tz = z$.

Therefore z is a common fixed point of T and g.

Let t be the another common fixed point of T and g.

Now consider, $\eta(\theta)\, p(gz, Tz) = \eta(\theta)\, p(z, z) = 0 \le p(gz, gt)$.

Hence from (2.1.1), we have

$$\begin{aligned} p(z,t) = p(Tz, Tt) &\le \theta \max\left\{ \frac{p(z,z)\, p(t,t)}{1 + p(z,t)}, \quad p(z,t) \right\} \\ &\le \theta\, \text{p(z,t)} < \text{p(z,t)}. \end{aligned}$$

Is a contradiction.

Hence z is a unique common fixed point of T and g.

Case(ii):If $gx_{n+1} = gx_n$ for some n, then $gx_n = Tx_n$.

That is $gu = Tu = z$, where $u = x_n$.

Since (T, g) is weakly compatible, we have $gz = Tz$.

$\eta(\theta)\, p(gu, Tu) = \eta(\theta)\, p(gu, gu) \le p(gu, gu) \le p(gz, gu)$, from (p_2)

From (2.1.1),it follows that

$$\begin{aligned} p(Tz, z) = p(Tz, Tu) &\le \theta \max\left\{ \frac{p(gz, Tz)\, p(z, z)}{1 + p(gz, z)}, \quad p(gz, z) \right\} \\ &= \theta \max\left\{ \frac{p(z, Tz)\, p(z, Tz)}{1 + p(z, Tz)}, p(Tz, z) \right\}, \text{from } (p_2) \\ &= \theta\ p(z, Tz) \end{aligned}$$

Thus gz = Tz = z.
Thus z is a common fixed point of g and T .
Let w be another common fixed point of g and T .
Since
$\eta(\theta)p(gz,Tz) \le p(gz,gz) \le p(gz,gw)$, from (p_2)

it follows from (2.1.1), that

$$p(z,w) = p(Tz,Tw) \le \theta \max\left\{\frac{p(z,z)p(w,w)}{1+p(z,w)}, \quad p(z,w)\right\}$$
$$\le \theta \max\left\{\frac{p(z,w)p(z,w)}{1+p(z,w)}, p(z,w)\right\}, \text{from } (p_2)$$
$$\le \theta\, p(z,w) < p(z,w).$$

It is a contradiction. Hence z = w.
Thus z is a unique common fixed point of T and g.
Example 2.2. Let X = [0, 1] and $p(x,y) = \max\{x, y\}$. Define

$$\text{T, g : X} \to \text{X by } Tx = \frac{x^2}{8};\; gx = \begin{cases} x & if \quad x \ne 1 \\ \frac{1}{2} & if \quad x = 1 \end{cases}$$

Then $T(X) = \left[0, \frac{1}{8}\right]$ is complete and T(X) $\subseteq$ g(X).

Let $\theta = \frac{3}{4}$. Then $\eta(\theta) = \frac{4}{7}$.

Case(i): If x $\ne$ 1, y $\ne$ 1

$$\eta(\theta)p(gx,Tx) = \frac{4}{7}\max\left\{x, \quad \frac{x^2}{8}\right\} \le p(gx,gy).$$

Now,

$$p(Tx,Ty) = \max\left\{\frac{x^2}{8}, \quad \frac{y^2}{8}\right\} \le \frac{3}{4}\max\{x, y\} = \frac{3}{4}p(gx,gy)$$
$$\le \theta \max\left\{\frac{p(gx,Tx)p(gy,Ty)}{1+p(gx,gy)}, \quad p(gx,gy)\right\}.$$

Case(ii): If x $\ne$ 1, y = 1.

$$\eta(\theta)p(gx,Tx) = \frac{4}{7}\max\left\{x, \quad \frac{x^2}{8}\right\} \le p(gx,gy).$$

Now,

$$p(Tx,Ty) = \max\left\{\frac{x^2}{8}, \ \frac{1}{8}\right\} \le \frac{3}{4}\max\left\{x, \ \frac{1}{2}\right\} = \frac{3}{4}p(gx,gy)$$
$$\le \theta \max\left\{\frac{p(gx,Tx)p(gy,Ty)}{1+p(gx,gy)}, \ p(gx,gy)\right\}.$$

Case(iii): If x = 1, y $\neq 1$.

$$\eta(\theta)p(gx,Tx) = \frac{4}{7}\max\left\{\frac{1}{2}, \ \frac{1}{8}\right\} \le p(gx,gy)$$

Now,

$$p(Tx,Ty) = \max\left\{\frac{1}{8}, \ \frac{y^2}{8}\right\} \le \frac{3}{4}\max\left\{y, \ \frac{1}{2}\right\} = \frac{3}{4}p(gx,gy)$$
$$\le \theta \max\left\{\frac{p(gx,Tx)p(gy,Ty)}{1+p(gx,gy)}, \ p(gx,gy)\right\}.$$

Case(iv): If x = 1, y =1.

$$\eta(\theta)p(gx,Tx) = \frac{4}{7}\max\left\{\frac{1}{2}, \ \frac{1}{8}\right\} \le p(gx,gy)$$

Now ,

$$p(Tx,Ty) = \max\left\{\frac{1}{8}, \ \frac{1}{8}\right\}$$
$$\le \frac{3}{8} = \frac{3}{4}p(gx,gy)$$
$$\le \theta \max\left\{\frac{p(gx,Tx)p(gy,Ty)}{1+p(gx,gy)}, \ p(gx,gy)\right\}.$$

Hence all conditions of Theorem 2.1 are satisfied and 0 is a unique common fixed point of T and g.

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