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On Star Numbers

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ABSRTACT

In this paper we report on the general form of the rank of square star numbers. The recurrence relations satisfied by the solutions are also found.

Keywords: The rank of square star numbers, the recurrence relations satisfied by the solutions.

Subject Classifications: MSC: 11A, 11D

INTRODUCTION:

The theory of numbers has been called the Queen of Mathematics because of its rich varieties of fascinating problems. Many numbers exhibit fascinating problems. Many numbers exhibit fascinating properties, they form sequences, they form patterns and so on.

A star number is a centered figurate number that represents a centered hexagram (six-pointed star), such as the one that Chinese checkers is played on. The nth star number is given by the formula $S_n = 6n(n-1)+1$. The first few star numbers are 1, 13 37, 73, 121, 181, …….The digital root of a star number is always 1 or 4, and progresses in the sequence 1, 4, 1. The last two digits of a star number in base 10 are always 01, 13, 21, 33, 37, 41, 53, 61, 73, 81, or 93.Unique among the star numbers is 35113, since its prime factors (i.e. 13, 37 and 73) are also consecutive star numbers.

Geometrically, the *n*th star number is made up of a central point and 12 copies of the $(n-1)^{th}$ triangular number — making it numerically equal to the *n*th centered dodecagonal number, but differently arranged. Infinitely many star numbers are also triangular numbers, the first four being $S_1 = 1 = T_1$, $S_7 = 253 = T_{22}$, $S_{91} = 49141 = T_{313}$, and $S_{1261} = 9533161 = T_{4366}$

S_n is given by the formula $S_n = 6n(n-1) + 1$ for $n \geq 1$

We determine the general form of the rank of square star numbers. Also the recurrence relations satisfied by the solutions are presented.

Theorem: The general form of the ranks of square star numbers (S_m) are given by

$$m = \frac{1}{4\sqrt{6}}\left(\left(5+2\sqrt{6}\right)^n\left(\sqrt{6}+2\right)+\left(5-2\sqrt{6}\right)^n\left(\sqrt{6}-2\right)\right)+\frac{1}{2}, n \geq 1$$

Proof: Let S_m be a square star number. We write

$S_m = t^2$, where t is a non-zero integer (1)

Using the definition of Star number the above equation (1) is written as

$$m^2 - m + \frac{1}{6} = \frac{t^2}{6}$$

By writing complete square and simplifying, we get

$$3(2m-1)^2 - 1 = 2t^2 \ldots \quad (2)$$

If we take x = 2m-1 then we get

$$3x^2 - 2t^2 = 1 \quad (3)$$

Which is the well –known Pell's equation whose solutions are given by

$$x_n = \frac{1}{2\sqrt{6}}\left(\left(5+2\sqrt{6}\right)^{n+1}\left(\sqrt{6}+2\right)+\left(5-2\sqrt{6}\right)^{n+1}\left(\sqrt{6}-2\right)\right) \quad (4)$$

$$t_n = \frac{1}{2\sqrt{6}}\left(\left(5+2\sqrt{6}\right)^{n+1}\left(\sqrt{6}+3\right)+\left(5-2\sqrt{6}\right)^{n+1}\left(\sqrt{6}-3\right)\right) \quad (5)$$

Where n = 0, 1, 2, ……

In view of the equation (2), the rank m of square Star number and the values of t are given by

$$m = \frac{1}{4\sqrt{6}}\left(\left(5+2\sqrt{6}\right)^n\left(\sqrt{6}+2\right)+\left(5-2\sqrt{6}\right)^n\left(\sqrt{6}-2\right)\right)+\frac{1}{2} \quad (6)$$

$$t = \frac{1}{2\sqrt{6}}\left(\left(5+2\sqrt{6}\right)^n\left(\sqrt{6}+3\right)+\left(5-2\sqrt{6}\right)^n\left(\sqrt{6}-3\right)\right) \quad (7)$$

where n =0, 1, 2, ……

For simplicity and brevity some values of m, t and their corresponding star and square numbers are presented in the following table.

Values of n	Ranks (m)	Ranks (t)	Star numbers (S_m)	Square numbers (t^2)
0	1	1	1	1
1	5	11	121	121
2	45	109	11881	11881
3	441	1079	1164241	1164241
4	4361	10681	114083761	114083761

Theorem: The ranks m_n and t_n of the equations (6) and (7) satisfy the following recurrence relations:

$$(i)\ m_{n+2} - 10m_{n+1} + m_n + 4 = 0$$

$$(ii)\ t_{n+2} - 10t_{n+1} + t_n = 0$$

PROOF:

Let $m = \frac{1}{4\sqrt{6}}\left(\left(5+2\sqrt{6}\right)^n\left(\sqrt{6}+2\right)+\left(5-2\sqrt{6}\right)^n\left(\sqrt{6}-2\right)\right)+\frac{1}{2}$ (8)

This equation (8) can be written as

$$4\sqrt{6}\,m_n - 2\sqrt{6} = \left(\left(5+2\sqrt{6}\right)^n\left(\sqrt{6}+2\right)+\left(5-2\sqrt{6}\right)^n\left(\sqrt{6}-2\right)\right) \quad (9)$$

If we take $A = 5+2\sqrt{6}\,, then\ A-4\sqrt{6} = 5-2\sqrt{6}$, $B = \sqrt{6}+2$, $then\ B-4 = \sqrt{6}-2$ and the equation (9) becomes

$$4\sqrt{6}\,m_n - 2\sqrt{6} = \left(A^nB+\left(A-4\sqrt{6}\right)^n(B-4)\right) \quad (10)$$

Replacing n by n+1, n+2 successively in (10), we get

$$2\sqrt{6}(2m_{n+1}-1) = \left(A^{n+1}B+\left(A-4\sqrt{6}\right)^{n+1}(B-4)\right) \quad (11)$$

$$2\sqrt{6}(2m_{n+2}-1) = \left(A^{n+2}B + \left(A-4\sqrt{6}\right)^{n+2}(B-4)\right) \quad (12)$$

Multiplying the equation (10) by A and then subtracting from the equation (11) we get

$$(2m_n - 1)A - 2m_{n+1} + 1 = 2\left(A-4\sqrt{6}\right)^n (B-4) \quad (13)$$

Multiplying the equation (11) by A and then subtracting from the equation (12) we get

$$(2m_{n+1} - 1)A - 2m_{n+2} + 1 = 2\left(A-4\sqrt{6}\right)^{n+1} \text{(B-4)} \quad (14)$$

Multiplying the equation (13) by $A-4\sqrt{6}$ and subtracting from the equation (14) we get,

$$(2m_{n+1}-1)A - 2m_{n+2} + 1 - (2m_n - 1)A\left(A-4\sqrt{6}\right) - 2m_{n+1}\left(A-4\sqrt{6}\right) + \left(A-4\sqrt{6}\right) = 0$$

On simplifying, we get

$$-2m_{n+2} + 20\, m_{n+1} - 2m_n - 8 = 0$$

On dividing by (- 2), we get

$$m_{n+2} - 10\, m_{n+1} + m_n + 4 = 0 \quad (15)$$

This is the recurrence relation satisfied by the ranks m.It is observed that the values of m: (1, 5, 45), (5, 45, 441) and (45, 441, 4361) are satisfied by the equation (15)

Proof of (ii)

Let $t_n = \frac{1}{2\sqrt{6}}\left(\left(5+2\sqrt{6}\right)^n\left(\sqrt{6}+3\right) + \left(5-2\sqrt{6}\right)^n\left(\sqrt{6}-3\right)\right)$ (16)

This equation (16) can be written as

$$2\sqrt{6}t_n = \left(\left(5+2\sqrt{6}\right)^n\left(\sqrt{6}+3\right) + \left(5-2\sqrt{6}\right)^n\left(\sqrt{6}-3\right)\right) \quad (17)$$

If we take $A = 5+2\sqrt{6}\,, then\ A-4\sqrt{6} = 5-2\sqrt{6},\ C = \sqrt{6}+3,\ then\ C-6 = \sqrt{6}-3$ and the equation (17) becomes

$$2\sqrt{6}t_n = \left((A)^n(C) + \left(A - 4\sqrt{6}\right)^n(C - 6)\right) \quad (18)$$

Replacing n by n+1, n+2 successively in (18), we get

$$2\sqrt{6}t_{n+1} = \left((A)^{n+1}(C) + \left(A - 4\sqrt{6}\right)^{n+1}(C - 6)\right) \quad (19)$$

$$2\sqrt{6}t_{n+2} = \left((A)^{n+2}(C) + \left(A - 4\sqrt{6}\right)^{n+2}(C - 6)\right) \quad (20)$$

Multiplying the equation (18) by A, and then subtracting from the equation (19) we get

$$\mathrm{At_n} - t_{n+1} = 2\left(A - 4\sqrt{6}\right)^n (C - 6) \quad (21)$$

Multiplying the equation (18) by A, and then subtracting from the equation (20) we get

$$\mathrm{At_{n+1}} - t_{n+2} = 2\left(A - 4\sqrt{6}\right)^n (C - 6) \quad (22)$$

Multiplying the equation (21) by$A - 4\sqrt{6}$, and then subtracting from the equation (22) we get

$$\left(A - 4\sqrt{6}\right)(At_n - t_{n+1}) - (At_{n+1} - t_{n+2}) = 0 \quad (23)$$

On simplifying, we get

$$t_{n+2} - 10t_{n+1} + t_n = 0 \quad (24)$$

This is the recurrence relation satisfied by the ranks t. It is observed that the values of t (1, 11, 109); (11, 109, 1079); (109, 1079, 10681) are satisfied by the equation (24)

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