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On the degenerate Frobenius-Euler polynomials

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Abstract

In this paper, we consider the degenerate Frobenius-Euler polynomials and investigate some identities of those polynomials.

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1. Introduction

For $u \in \mathbb{C}$ with $u \neq 1$, the *Frobenius-Euler polynomials* are defined by the generating function to be

$$\frac{1-u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x|u) \frac{t^n}{n!}, \text{ (see [1-10]).} \tag{1.1}$$

When $x = 0$, $H_n(x|u) = H_n(0|u)$ are called the *Frobenius-Euler numbers*.

From (1.1), we have

$$H_n(x|u) = \sum_{l=0}^{n} \binom{n}{l} H_l(u) x^{n-l}, \ (n \geq 0), \ \text{(see [6-8])}. \tag{1.2}$$

Note that

$$\frac{d}{dx} H_n(x|u) = n H_{n-1}(x|u), \ (n \in \mathbb{N}).$$

In [3], L. Carlitz define the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda) \frac{t^n}{n!}. \tag{1.3}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(0|\lambda)$ are called the *degenerate Bernoulli numbers*. From (1.3), we have

$$\lim_{\lambda \to 0} \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}} = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.4}$$

where $B_n(x)$ are called *Bernoulli polyomials*

By (1.3) and (1.4), we get

$$\lim_{\lambda \to 0} \beta_n(x|\lambda) = B_n(x), \ (n \geq 0).$$

The *Stirling number of the first kind* is defined as

$$(x)_n = \sum_{l=0}^{n} S_1(n,l) x^l, \ (n \geq 0), \tag{1.5}$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, $(x)_0 = 1$.

By (1.3), we easily get

$$\beta_n(x|\lambda) = \sum_{l=0}^{n} \binom{n}{l} \beta_{n-l}(x|\lambda)(x|\lambda)_l, \ (n \geq 0), \tag{1.6}$$

where $(x|\lambda)_n = x(x-\lambda)\cdots(x-(n-1)\lambda)$.

The *Stirling numbers of the second kind* is defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \ (n \geq 0). \tag{1.7}$$

In this paper, we consider the degenerate Frobenius-Euler polynomials and investigate some properties and identities of those polynomials.

2. Degenerate Frobenius-Euler polynomials

For $u \in$ with $u \neq 1$, we consider degenerate Frobenius-Euler polynomials which are given by the generating function to be

$$\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} h_{n,\lambda}(x|u)\frac{t^n}{n!}. \tag{2.1}$$

When $x=0$, $h_{n,\lambda}(u) = h_{n,\lambda}(0|u)$ are called *degenerate Frobenius-Euler numbers*.

From (2.1), we have

$$\begin{aligned}\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}(1+\lambda t)^{\frac{x}{\lambda}} &= \left(\sum_{l=0}^{\infty}\frac{h_{l,\lambda}(u)}{l!}t^n\right)\left(\sum_{m=0}^{\infty}(x|\lambda)_m\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}h_{l,\lambda}(u)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

Thus by (2.1) and (2.2), we get

$$h_{n,\lambda}(x|u) = \sum_{l=0}^{n}\binom{n}{l}h_{l,\lambda}(u)(x|\lambda)_{n-l}. \tag{2.3}$$

From (2.1), we can derive the following recurrence relation:

$$\begin{aligned}1-u =&(1+\lambda t)^{\frac{1}{\lambda}}\sum_{n=0}^{\infty}h_{n,\lambda}(u)\frac{t^n}{n!} - u\sum_{n=0}^{\infty}h_{n,\lambda}(u)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}h_{n-l,\lambda}(u)(1|\lambda)_l\right)\frac{t^n}{n!} - u\sum_{n=0}^{\infty}h_{n,\lambda}(u)\frac{t^n}{n!}\\ =&\sum_{n=0}^{\infty}\left(h_{n,\lambda}(1|u) - uh_{n,\lambda}(u)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

By comparing of the coefficients on the both sides of (2.4), we get

$$h_{n,\lambda}(1|u) - uh_{n,\lambda}(u) = \begin{cases}1-u, & \text{if } n=0\\ 0, & \text{if } n>0.\end{cases} \tag{2.5}$$

Note that

$$\lim_{\lambda\to 0}\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}(1+\lambda t)^{\frac{x}{\lambda}} = \frac{1-u}{e^t-u}e^{xt} = \sum_{n=0}^{\infty}H_n(x|u)\frac{t^n}{n!}. \tag{2.6}$$

From (2.1) and (2.6), we have

$$H_n(x|u) = \lim_{\lambda\to 0} h_{n,\lambda}(x|u), \ (n \geq 0). \tag{2.7}$$

We observe that

$$\begin{aligned} &\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}(1+\lambda t)^{\frac{x+1}{\lambda}} - \frac{u(1-u)}{(1+\lambda t)^{\frac{1}{\lambda}}-u}(1+\lambda t)^{\frac{x}{\lambda}} \\ =&(1-u)(1+\lambda t)^{\frac{x}{\lambda}} = (1-u)\sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \end{aligned} \tag{2.8}$$

Therefore, by (2.1) and (2.8), we get

$$h_{n,\lambda}(x+1|u) - uh_{n,\lambda}(x|u) = (x|\lambda)_n(1-u), \ (n \geq 0). \tag{2.9}$$

Therefore, we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$h_{n,\lambda}(x|u) = \sum_{l=0}^{n}\binom{n}{l}h_{l,\lambda}(u)(x|\lambda)_{n-l}$$

and

$$\frac{h_{n,\lambda}(x+1|u)}{1-u} - \frac{u}{1-u}h_{n,\lambda}(x|u) = (x|\lambda)_n.$$

In particular,

$$h_{n,\lambda}(1|u) - uh_{n,\lambda}(u) = (1-u)\delta_{0,n},$$

where $\delta_{n,k}$ is the Kronecker's symbol.

By (2.1), we get

$$\begin{aligned} \sum_{n=0}^{\infty}h_{n,-\lambda}(u)\frac{t^n}{n!} =&\frac{1-u}{(1-\lambda t)^{-\frac{1}{\lambda}}-u} = \frac{1-u}{1-u(1-\lambda t)^{\frac{1}{\lambda}}}(1-\lambda t)^{\frac{1}{\lambda}} \\ =&\frac{1-u^{-1}}{(1-\lambda t)^{\frac{1}{\lambda}}-u^{-1}}(1-\lambda t)^{\frac{1}{\lambda}} = \sum_{n=0}^{\infty}h_{n,\lambda}(1|u^{-1})(-1)^n\frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

In general,

$$\begin{aligned} \sum_{n=0}^{\infty}h_{n,-\lambda}(-x|u)\frac{t^n}{n!} =&\frac{1-u}{(1-\lambda t)^{-\frac{1}{\lambda}}-u}(1-\lambda t)^{\frac{x}{\lambda}} = \frac{1-u^{-1}}{(1-\lambda t)^{\frac{1}{\lambda}}-u^{-1}}(1-\lambda t)^{\frac{x+1}{\lambda}} \\ =&\sum_{n=0}^{\infty}h_{n,\lambda}(x+1|u^{-1})(-1)^n\frac{t^n}{n!}. \end{aligned} \tag{2.11}$$

Therefore, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$(-1)^n h_{n,-\lambda}(-x|u) = h_{n,\lambda}(x+1|u^{-1}).$$

In particular,

$$(-1)^n h_{n,-\lambda}(u) = (-1)^n h_{n,\lambda}(1|u^{-1}).$$

We observe that

$$\begin{aligned} -\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}(1+\lambda t)^{\frac{x}{\lambda}} =& \frac{u^d(1-u^{-1})}{(1+\lambda t)^{\frac{d}{\lambda}}-u^d}\sum_{l=0}^{d-1} u^{-l}(1+\lambda t)^{\frac{l+x}{\lambda}} \\ =& \frac{u^d(1-u^{-1})}{1-u^d}\sum_{l=0}^{d-1}\frac{u^{-l}(1-u^d)}{(1+\lambda t)^{\frac{d}{\lambda}}-u^d}(1+\lambda t)^{\frac{l+x}{\lambda}} \\ =& \frac{(1-u^{-1})}{1-u^d}\sum_{l=0}^{d-1}\sum_{n=0}^{\infty} u^{d-l} h_{n,\frac{\lambda}{d}}\left(\frac{l+x}{d}\middle| u^d\right) d^n \frac{t^n}{n!} \\ =& \sum_{n=0}^{\infty}\left\{\left(\frac{1-u^{-1}}{1-u^d}\right) d^n \sum_{a=0}^{d-1} u^{d-a} h_{n,\frac{\lambda}{d}}\left(\frac{a+x}{d}\middle| u^d\right)\right\}\frac{t^n}{n!}. \end{aligned} \tag{2.12}$$

Therefore, by (2.1) and (2.12), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, $d \in \mathbb{N}$, we have

$$h_{n,\lambda}(x|u) = d^n \left(\frac{-1+u^{-1}}{1-u^d}\right)\sum_{a=0}^{d-1} u^{d-a} h_{n,\frac{\lambda}{d}}\left(\frac{a+x}{d}\middle| u^d\right).$$

For $r \in \mathbb{N}$, let us consider the degenerate Frobenius-Euler polynomials of order r which are given by the generating function to be

$$\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^r (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} h_{n,\lambda}^{(r)}(x|u)\frac{t^n}{n!}. \tag{2.13}$$

When $x=0$, $h_{n,\lambda}^{(r)}(u) = h_{n,\lambda}^{(r)}(0|u)$ are called *degenerate Frobenius-Euler numbers of order* r.

From (2.13), we can easily derive the following equation.

$$h_{n,\lambda}^{(r)}(x) = \sum_{l=0}^{n}\binom{n}{l} h_{l,\lambda}^{(r)}(u)(x|\lambda)_{n-l}, \ (n \geq 0). \tag{2.14}$$

By (2.13), we get

$$\begin{aligned}\sum_{n=0}^{\infty} h_{n,\lambda}^{(r)}(x+y|\lambda)\frac{t^n}{n!} &= \left(\sum_{l=0}^{\infty} h_{l,\lambda}^{(r)}(x)\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}(x|\lambda)_m\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} h_{l,\lambda}^{(r)}(x)(x|\lambda)_{n-l}\binom{n}{l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.15}$$

Therefore, by (2.15), we obtain the following theorem.

Theorem 2.4. For $n \geq 0$, we have

$$h_{n,\lambda}^{(r)}(x+y)|\lambda) = \sum_{l=0}^{n}\binom{n}{l} h_{l,\lambda}^{(r)}(x)(x|\lambda)_{n-l}.$$

From Theorem 2.4, we note that $h_{n,\lambda}^{(r)}(x|u)$ is a Sheffer sequence.

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.13), we get

$$\begin{aligned}\left(\frac{1-u}{e^t-u}\right)^r e^{xt} &= \sum_{n=0}^{\infty} h_{n,\lambda}^{(r)}(x|u)\frac{1}{n!}\left(\frac{1}{\lambda}(e^{\lambda t}-1)\right)^n \\ &= \sum_{n=0}^{\infty} h_{n,\lambda}^{(r)}(x|u)\lambda^{-n}\sum_{m=n}^{\infty} S_2(m,n)\lambda^m\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} h_{n,\lambda}^{(r)}(x|u)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.16}$$

As is well known, the higher-order Frobenius-Euler polynomials are defined by the generating function to be

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|u)\frac{t^n}{n!}. \tag{2.17}$$

Thus, by (2.16) and (2.17), we get

$$H_n^{(r)}(x|u) = \sum_{n=0}^{m} h_{n,\lambda}(x|u)\lambda^{m-n}S_2(m,n). \tag{2.18}$$

Therefore, by (2.18), we obtain the following theorem.

Theorem 2.5. For $m \geq 0$, we have

$$H_n^{(r)}(x|u) = \sum_{n=0}^{m} h_{n,\lambda}^{(r)}(x|u)\lambda^{m-n}S_2(m,n).$$

By replace t by $\log(1+\lambda t)^{\frac{1}{\lambda}}$ in (2.17), we have

$$\begin{aligned}\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^r(1+\lambda t)^{\frac{x}{\lambda}} &= \sum_{n=0}^{\infty} H_n^{(r)}(x|u)\frac{1}{n!}\frac{1}{\lambda^n}\left(\log(1+\lambda t)\right)^n \\ &= \sum_{n=0}^{\infty} H_n^{(r)}(x|u)\lambda^{-n}\sum_{m=n}^{\infty} S_1(m,n)\lambda^m\frac{t^n}{n!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} H_n^{(r)}(x|u)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.19}$$

Therefore, by (2.13) and (2.18), we obtain the following theorem.

Theorem 2.6. For $m \geq 0$, we have

$$h_{n,\lambda}^{(r)}(x|u) = \sum_{n=0}^{m} H_n^{(r)}(x|u)\lambda^{m-n}S_1(m,n).$$

We observe that

$$\begin{aligned}&\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^r(1+\lambda t)^{\frac{x+1}{\lambda}} - u\left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^r(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \left(\frac{1-u}{(1+\lambda t)^{\frac{1}{\lambda}}-u}\right)^{r-1}(1+\lambda t)^{\frac{x}{\lambda}}(1-u) = \sum_{n=0}^{\infty} h_{n,\lambda}^{(r-1)}(x|u)(1-u)\frac{t^n}{n!}.\end{aligned} \tag{2.20}$$

Therefore, by (2.13), we obtain the following theorem.

Theorem 2.7. For $n \geq 0$, we have

$$\frac{1}{1-u}\left\{h_{n,\lambda}^{(r)}(x+1|u) - u h_{n,\lambda}^{(r)}(x|u)\right\} = h_{n,\lambda}^{(r-1)}(x|u).$$

Remark. 2.8. et $u=-1$. Then, by (2.1), we get

$$\frac{1}{t}\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} h_{n,\lambda}(x|-1)\frac{t^n}{n!}. \tag{2.21}$$

Thus, by (2.21), we get

$$\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = t\sum_{n=0}^{\infty} h_{n,\lambda}(x|-1)\frac{t^n}{n!}.$$

Now, we define the degenerate Genocchi polynomials which are given by the generating function to be

$$\frac{2t}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} g_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.22}$$

From (2.21) and (2.22), we have

$$g_{0,\lambda}(x) = 0 \text{ and } h_{n,\lambda}(x|-1) = \frac{1}{n+1} g_{n+1,\lambda}(x), \ (n \geq 0).$$

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