Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 4 (2015), pp. 2065-2076

http://www.ripublication.com

Some new separation Axioms in grill topological spaces

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Abstract

In this paper, $\mathcal{G}$-semiopen sets are used to define some weak separation axioms and to study some of their basic properties. The implications of these axioms among themselves and with the known axioms are investigated.

AMS subject classification: 54D10.
Keywords: Grill topological spaces, $\mathcal{G}$-semi-R_0 space, $\mathcal{G}$-semi-R_1 space, $\mathcal{G}$-semi-R_2 space.

1. Introduction

The notion of R_0 topological spaces is introduced by Shanin [1] in 1943. Later, Davis [2] rediscovered it and studied some properties of this weak separation axiom. Several topologists (e.g. [3], [4], [5]) further investigated properties of R_0 topological spaces and many interesting results have been obtained in various contexts. In the same paper, Davis also introduced the notion of R_1 topological space which are independent of both T_0 and T_1 but strictly weaker than T_2. The idea of grills on a topological space was first introduced by Choquet [6]. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds (see [7], [8], [9] for details). In [10], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. Quite recently, D. Mondal and Mukherjee [11] have defined new classes of sets in a grill topological space and obtained a new decomposition of con- tinuity in terms of grills. In this paper, $\mathcal{G}$-semiopen sets are used to define some weak separation axioms and to study some of their basic properties. The implications of these axioms among themselves and with the known axioms are investigated.

2. Preliminaries

Let (X, τ) be a topological space with no separation properties assumed. For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure and the interior of A in (X, τ), respectively. The power set of X will be denoted by $\mathcal{P}(X)$. The definition of grill on a topological space, as given by Choquet [6], goes as follows: A non-null collection $\mathcal{G}$ of subsets of a topological space (X, τ) is said to be a grill on X if

(1) $\emptyset \notin \mathcal{G}$

(2) $A \in \mathcal{G}$ and $A \subset B$ implies that $B \in \mathcal{G}$

(3) $A, B \subset X$ and $A \subset B \in \mathcal{G}$ implies that $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 2.1. [10] Let (X, τ) be a topological space and $\mathcal{G}$ a grill on X. A mapping $\Phi : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: $\Phi(A) = \Phi_{\mathcal{G}}(A, \tau) = \{x \in X : A \cap U \in \mathcal{G}$ for every open set U containing $x\}$ for each $A \in \mathcal{P}(X)$. The mapping Φ is called the operator associated with the grill $\mathcal{G}$ and the topology τ.

Definition 2.2. [10] Let $\mathcal{G}$ be a grill on a topological space (X, τ). Then we define a map $\Psi : \mathcal{P}(X) \to \mathcal{P}(X)$ by $\Psi(A) = A \cup \Phi(A)$ for all $A \in \mathcal{P}(X)$. The map Ψ is a Kuratowski closure axiom. Corresponding to a grill $\mathcal{G}$ on a topological space (X, τ), there exists a unique topology $\tau_{\mathcal{G}}$ on X given by $\tau_{\mathcal{G}} = \{U \subset X : \Psi(X\backslash U) = X\backslash U\}$, where for any $A \subset X$, $\Psi(A) = A \cup \Phi(A) = \tau_{\mathcal{G}}\ \mathrm{Cl}(A)$. For any grill $\mathcal{G}$ on a topological space (X, τ), $\tau \subset \tau_{\mathcal{G}}$. If (X, τ) is a topological space with a grill $\mathcal{G}$ on X, then we call it a grill topological space and denote it by $(X, \tau, \mathcal{G})$.

Definition 2.3. [11] A subset S of a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semiopen if $S \subset \Psi(\mathrm{Int}(S))$. The complement of a $\mathcal{G}$-semiopen set is called a $\mathcal{G}$-semiclosed set.

Definition 2.4. The intersection of all $\mathcal{G}$-semiclosed sets containing $S \subset X$ is called the $\mathcal{G}$-semiclosure of S and is denoted by $s\mathrm{Cl}_{\mathcal{G}}(S)$. The family of all $\mathcal{G}$-semiopen (resp. $\mathcal{G}$-semiclosed) sets of $(X, \tau, \mathcal{G})$ is denoted by $\mathcal{G}SO(X)$ (resp. $\mathcal{G}SC(X)$). The family of all $\mathcal{G}$-semiopen (resp. $\mathcal{G}$-semiclosed) sets of $(X, \tau, \mathcal{G})$ containing a point $x \in X$ is denoted by $\mathcal{G}SO(X, x)$ (resp. $\mathcal{G}SC(X, x)$).

Definition 2.5. A subset U of X is called a $\mathcal{G}$-semineighbourhood of a point $x \in X$ if there exists a $\mathcal{G}$-semiopen set V of $(X, \tau, \mathcal{G})$ such that $x \in V \subset U$.

Definition 2.6. [11] A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is said to be G-semicontinuous if $f^{-1}(V)$ is a $\mathcal{G}$-semiopen set in $(X, \tau, \mathcal{G})$ for every open set V of Y.

Definition 2.7. A grill topological space $(X, \tau, \mathcal{G})$ is said to be:

(1) $\mathcal{G}$-semi-T_1 [12] if for each pair of distinct points x and y of X, there exist $\mathcal{G}$-semiopen sets U and V of X such that $x \in U$ and $y \notin U$, and $y \in V$ and $x \notin V$.

(2) $\mathcal{G}$-semi-T_2 [12] if for each pair of distinct points x and y in X, there exist disjoint $\mathcal{G}$-semiopen sets U and V in X such that $x \in U$ and $y \in V$.

3. On $\mathcal{G}$-semi-R_0 spaces

Definition 3.1. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $A \subset X$. Then the $\mathcal{G}$-semikernel of A, denoted by $s\mathrm{Ker}_{\mathcal{G}}(A)$ is defined to be the set $s\mathrm{Ker}_{\mathcal{G}}(A) = \cap\{G \in \mathcal{G}SO(X) | A \subset G\}$.

Lemma 3.2. Let $(X, \tau, \mathcal{G})$ be a grill topological space and $x \in X$. Then, $y \in s\mathrm{Ker}_{\mathcal{G}}(\{x\})$ if and only if $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$.

Proof. Suppose that $y \notin s\mathrm{Ker}_{\mathcal{G}}(\{x\})$. Then there exists $U \in \mathcal{G}SO(X, x)$ such that $y \notin U$. Therefore, we have $x \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. The proof of the converse case can be done similarly. ∎

Lemma 3.3. Let $(X, \tau, \mathcal{G})$ be a grill topological space and A a subset of X. Then, $s\mathrm{Ker}_{\mathcal{G}}(A) = \{x \in X | s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap A \neq \varnothing\}$.

Proof. Let $x \in s\mathrm{Ker}_{\mathcal{G}}(A)$ and $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap A = \varnothing$. Hence $x \notin X \setminus s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ which is a $\mathcal{G}$-semiopen set containing A. This is impossible, since $x \in s\mathrm{Ker}_{\mathcal{G}}(A)$. Consequently, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap A \neq \varnothing$. Next, let $x \in X$ such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap A \neq \varnothing$ and suppose that $x \notin s\mathrm{Ker}_{\mathcal{G}}(A)$. Then, there exists a $\mathcal{G}$-semiopen set U containing A and $x \notin U$. Let $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap A$. Hence, U is a $\mathcal{G}$-semineighbourhood of y which does not contains x. By this contradiction $x \in s\mathrm{Ker}_{\mathcal{G}}(A)$ and hence the claim. ∎

Definition 3.4. A grill topological space $(X, \tau, \mathcal{G})$ is said to be a $\mathcal{G}$- semi-R_0 space if every $\mathcal{G}$-semiopen set contains the $\mathcal{G}$-semiclosure of each of its singletons.

Remark 3.5. Since a grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-T_1 if and only if the singletones are $\mathcal{G}$-semiclosed [12], it is clear that every $\mathcal{G}$-semi-T_1 space $\mathcal{G}$-semi-R_0. But the converse is not true in general.

Example 3.6. Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, \{b, c\}, X\}$ and $\mathcal{G} = \{\varnothing, \{a\}, \{a, b\}, \{a, c\}, X\}$. Then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_0 but none of $\mathcal{G}$-semi-T_0 and $\mathcal{G}$-semi-T_1.

Remark 3.7. The following example and Example 3.6 shows that the notions $\mathcal{G}$-semi-T_0-ness $\mathcal{G}$-semi-R_0-ness are independent.

Example 3.8. Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{b\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$. Then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-T_0 but not $\mathcal{G}$-semi-R_0.

Proposition 3.9. For a grill topological space $(X, \tau, \mathcal{G})$, the following properties are equivalent:

(1) $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_0 space;

(2) For any $F \in \mathcal{G}SC(X)$, $x \notin F$ implies $F \subset U$ and $x \notin U$ for some $U \in \mathcal{G}SO(X)$;

(3) For any $F \in \mathcal{G}SC(X)$, $x \notin F$ implies $F \cap s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = \varnothing$;

(4) For any distinct points x and y of X, either $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ or $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = \varnothing$.

Proof. (1) $\Rightarrow$ (2): Let $F \in \mathcal{G}SC(X)$ and $x \notin F$. Then by (1) $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset X\backslash F$. Set $U = X\backslash s\mathrm{Cl}_{\mathcal{G}}(\{x\})$, then $U \in \mathcal{G}SO(X)$, $F \subset U$ and $x \notin U$.
(2) $\Rightarrow$ (3): Let $F \in \mathcal{G}SC(X)$ and $x \notin F$. There exists $U \in \mathcal{G}SO(X)$ such that $F \subset U$ and $x \notin U$. Since $U \in \mathcal{G}SO(X)$, $U \cap s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = \varnothing$ and $F \cap s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = \varnothing$.
(3) $\Rightarrow$ (4): Suppose that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ for distinct points $x, y \in X$. There exists $z \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ such that $z \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ (or $z \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ such that $z \notin s\mathrm{Cl}_{G}(\{x\})$). There exists $V \in \mathcal{G}SO(X)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. By (3), we obtain $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = \varnothing$. The proof for otherwise is similar.
(4) $\Rightarrow$ (1): Let $V \in \mathcal{G}SO(X, x)$. For each $y \notin V$, $x \neq y$ and $x \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. This shows that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. By (4), $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = \varnothing$ for each $y \in X\backslash V$ and hence $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap (\cup_{y \in X\backslash V}\ s\mathrm{Cl}_{\mathcal{G}}(\{y\})) = \varnothing$. On other hand, since $V \in \mathcal{G}SO(X)$ and $y \in X\backslash V$, we have $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \subset X\backslash V$ and hence $X\backslash V = \cup_{y \in X\backslash V}\ s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Therefore, we obtain $(X\backslash V) \cap s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = \varnothing$ and $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset V$. This shows that $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space.

■

Theorem 3.10. A grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space if and only if for any x and y in X, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ implies $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = \varnothing$.

Proof. Suppose that $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_0 and $x, y \in X$ such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Then, there exists $z \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ such that $z \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ (or $z \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$) such that $z \notin s\mathrm{Cl}_{\mathcal{G}}(\{x\})$. There exists $V \in \mathcal{G}SO(X)$ such that $y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Thus $x \in X\backslash s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \in \mathcal{G}SO(X)$, which implies $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset X \backslash s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ and $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = \emptyset$. The proof for otherwise is similar. Conversely, let $V \in \mathcal{G}SO(X, x)$. We will show that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset V$. Let $y \in X \backslash V$. Then $x \neq y$ and $x \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. This shows that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. By assumption, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = \emptyset$. Hence $y \notin s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ and therefore $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset V$. ■

Lemma 3.11. The following statements are equivalent for any points x and y in a grill topological space $(X, \tau, \mathcal{G})$:

(1) $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Ker}_{\mathcal{G}}(\{y\})$;

(2) $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$.

Proof. (1) $\Rightarrow$ (2): Suppose that $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Ker}_{\mathcal{G}}(\{y\})$, then there exists a point z in X such that $z \in s\mathrm{Ker}_{\mathcal{G}}(\{x\})$ and $z \notin s\mathrm{Ker}_{\mathcal{G}}(\{y\})$. It follows from $z \in s\mathrm{Ker}_{\mathcal{G}}(\{x\})$ that $\{x\} \cap s\mathrm{Cl}_{\mathcal{G}}(\{z\}) \neq \emptyset$. This implies that $x \in s\mathrm{Cl}_{\mathcal{G}}(\{z\})$. By $z \notin s\mathrm{Ker}_{\mathcal{G}}(\{y\})$, we have $\{y\} \cap s\mathrm{Cl}_{\mathcal{G}}(\{z\}) = \emptyset$. Since $x \in s\mathrm{Cl}_{\mathcal{G}}(\{z\})$, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset s\mathrm{Cl}_{\mathcal{G}}(\{z\})$ and $\{y\} \cap s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = \emptyset$. Therefore, it follows that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Now $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Ker}_{\mathcal{G}}(\{y\})$ implies that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$.
(2) $\Rightarrow$ (1): Suppose that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Then there exists a point z in X such that $z \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ and $z \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Then, there exists a $\mathcal{G}$-semiopen set containing z and therefore x but not y, namely, $y \notin s\mathrm{Ker}_{\mathcal{G}}(\{x\})$ and thus $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Ker}_{\mathcal{G}}(\{y\})$. ■

Theorem 3.12. A grill topological space $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space if and only if for any pair of points x and y in X, $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Ker}_{\mathcal{G}}(\{y\})$ implies $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Ker}_{\mathcal{G}}(\{y\}) = \emptyset$.

Proof. Suppose that $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space. Thus by Lemma 3.11, for any points x and y in X if $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Ker}_{\mathcal{G}}(\{y\})$, then $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Now we prove that $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Ker}_{\mathcal{G}}(\{y\}) = \emptyset$. Assume that $z \in s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Ker}_{\mathcal{G}}(\{y\})$. By $z \in s\mathrm{Ker}_{\mathcal{G}}(\{x\})$ and Lemma 3.11, it follows that $x \in s\mathrm{Cl}_{\mathcal{G}}(\{z\})$. Since $x \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$, by Theorem 3.10 $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Cl}_{\mathcal{G}}(\{z\})$. Similarly, we have $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = s\mathrm{Cl}_{\mathcal{G}}(\{z\}) = s\mathrm{Cl}_{\mathcal{G}}(\{x\})$. This is a contradiction. Therefore, we have $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Ker}_{\mathcal{G}}(\{y\}) = \emptyset$. Conversely, let $(X, \tau, \mathcal{G})$ be a grill topological space such that for any points x and y in X, $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Ker}_{\mathcal{G}}(\{y\})$ implies $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Ker}_{\mathcal{G}}(\{y\}) = \emptyset$. If $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, then by Lemma 3.2, $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Ker}_{\mathcal{G}}(\{y\})$. Hence, $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Ker}_{\mathcal{G}}(\{y\}) = \emptyset$ which implies $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = \emptyset$. Because $z \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ implies that $x \in s\mathrm{Ker}_{\mathcal{G}}(\{z\})$ and therefore $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Ker}_{\mathcal{G}}(\{y\}) \neq \emptyset$. By hypothesis, we have $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) = s\mathrm{Ker}_{\mathcal{G}}(\{z\})$. Then $z \in s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ implies that $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) = s\mathrm{Ker}_{\mathcal{G}}(\{z\}) = s\mathrm{Ker}_{\mathcal{G}}(\{y\})$. This is a contradiction. Therefore, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = \emptyset$ and by Theorem 3.10 $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space. ■

Theorem 3.13. For a grill topological space $(X, \tau, \mathcal{G})$, the following properties are equivalent:

(1) $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space;

(2) For any nonempty sets A, $\mathcal{G} \in \mathcal{G}SO(X)$ such that $A \cap \mathcal{G} \neq \varnothing$, there exists $F \in \mathcal{G}SC(X)$ such that $A \cap F \neq \varnothing$ and $F \subset G$;

(3) Any $G \in \mathcal{G}SO(X), \mathcal{G} = \cup\{F \in \mathcal{G}SC(X) | F \subset G\}$;

(4) Any $F \in \mathcal{G}SC(X)$, $F = \cap\{G \in \mathcal{G}SO(X) | F \subset G\}$;

(5) For any $x \in X$, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset s\mathrm{Ker}_{\mathcal{G}}(\{x\})$.

Proof. (1) $\Rightarrow$ (2): Let A be a nonempty set of X and $G \in \mathcal{G}SO(X)$ such that $A \cap G \neq \varnothing$. There exists $x \in A \cap G$. Since $x \in G \in \mathcal{G}SO(X)$, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset G$. Set $F = s\mathrm{Cl}_{\mathcal{G}}(\{x\})$, then $F \in \mathcal{G}SC(X)$, $F \subset G$ and $A \cap F \neq \varnothing$.
(2) $\Rightarrow$ (3): Let $G \in \mathcal{G}SO(X)$, then $G \supset \cup\{F \in \mathcal{G}SC(X) | F \subset G\}$. Let x be any point of G. There exists $F in \mathcal{G}SC(X)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \cup\{F \in \mathcal{G}SC(X) | F \subset G\}$ and hence $G = \cup\{F \in \mathcal{G}SC(X) | F \subset G\}$.
(3) $\Rightarrow$ (4): This is obvious.
(4) $\Rightarrow$ (5): Let x be any point of X and $y \notin s\mathrm{Ker}_{\mathcal{G}}(\{x\})$. There exists $V \in \mathcal{G}SO(X, x)$ $y \notin V$; hence $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \cap V = \varnothing$. By (4) $(\cap\{G \in \mathcal{G}SO(X) | s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \subset G\}) \cap V = \varnothing$ and there exists $G \in \mathcal{G}SO(X)$ such that $x \notin G$ and $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \subset G$. Therefore, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap G = \varnothing$ and $y \notin s\mathrm{Cl}_{\mathcal{G}}(\{x\})$. Consequently, we obtain $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset s\mathrm{Ker}_{\mathcal{G}}(\{x\})$.
(5) $\Rightarrow$ (1): Let $G \in \mathcal{G}SO(X, x)$. Let $y \in s\mathrm{Ker}_{\mathcal{G}}(\{x\})$, then $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ and $y \in G$. This implies that $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \subset G$. Therefore, we obtain $x \in s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \subset G$. This shows that $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space. ∎

Corollary 3.14. For a grill topological space $(X, \tau, \mathcal{G})$, the following properties are equivalent:

(1) $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space;

(2) $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Ker}_{\mathcal{G}}(\{x\})$ for all $x \in X$.

Proof. (1) $\Rightarrow$ (2): Suppose that $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space. By Theorem 3.13, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset s\mathrm{Ker}_{\mathcal{G}}(\{x\})$ for each $x \in X$. Let $y \in s\mathrm{Ker}_{\mathcal{G}}(\{x\})$, then $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ and by Theorem 3.10 $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Therefore, $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ and hence $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \subset s\mathrm{Cl}_{\mathcal{G}}(\{x\})$. This shows that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Ker}_{\mathcal{G}}(\{x\})$.
(2) $\Rightarrow$ (1): This is obvious by Theorem 3.13. ∎

Corollary 3.15. If for any point x of a $\mathcal{G}$-semi-R_0 space $(X, \tau, \mathcal{G})$, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Ker}_{\mathcal{G}}(\{x\}) = \{x\}$, then $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) = \{x\}$.

Proof. The proof follows from Theorem 3.13(v). ∎

Theorem 3.16. For a grill topological space $(X, \tau, \mathcal{G})$, the following properties are equivalent:

(1) $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space;

(2) $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ if and only if $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ for any points x and y in X.

Proof. (1) $\Rightarrow$ (2): Assume that $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_0 . Let $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ and $A \in \mathcal{G}SO(X, y)$. Now by hypothesis, $x \in A$. Therefore, every $\mathcal{G}$-semiopen set containing y contains x. Hence $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$.
(2) $\Rightarrow$ (1): Let $U \in \mathcal{G}SO(X, x)$. If $y \notin U$, then $x \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ and hence $y \notin s\mathrm{Cl}_{\mathcal{G}}(\{x\})$. This implies that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset U$. Hence $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_0. ■

Theorem 3.17. For a grill topological space $(X, \tau, \mathcal{G})$, the following properties are equivalent:

(1) $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space;

(2) If F is a $\mathcal{G}$-semiclosed subset of X, then $F = s\mathrm{Ker}_{\mathcal{G}}(F)$;

(3) If F is a $\mathcal{G}$-semiclosed subset of X and $x \in F$, then $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \subset F$;

(4) If $x \in X$, then $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \subset s\mathrm{Cl}_{\mathcal{G}}(\{x\})$.

Proof. (1) $\Rightarrow$ (2): Let F be $\mathcal{G}$-semiclosed subset of X and $x \notin F$. Thus $X \backslash F \in \mathcal{G}SO(X, x)$. Since $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_0, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset X \backslash F$. Thus $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap F = \varnothing$ and Lemma 3.3 $x \notin s\mathrm{Ker}_{\mathcal{G}}(F)$. Therefore, $s\mathrm{Ker}_{\mathcal{G}}(F) = F$.
(2) $\Rightarrow$ (3): In general, $A \subset B$ implies $s\mathrm{Ker}_{\mathcal{G}}(A) \subset s\mathrm{Ker}_{\mathcal{G}}(B)$. Therefore, it follows from (2) thats $\mathrm{Ker}_{\mathcal{G}}(\{x\}) \subset s\mathrm{Ker}_{\mathcal{G}}(F) = F$.
(3) $\Rightarrow$ (4): Since $x \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ and $s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ is $\mathcal{G}$-semiclosed, by (3) $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \subset s\mathrm{Cl}_{\mathcal{G}}(\{x\})$.
(4) $\Rightarrow$ (1): We show the implication by using Theorem 3.16. Let $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Then by Lemma 3.2 $y \in s\mathrm{Ker}_{\mathcal{G}}(\{x\})$. Since $x \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ and $s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ is $\mathcal{G}$-semiclosed, by (4) we obtain $y \in s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \subset s\mathrm{Cl}_{\mathcal{G}}(\{x\})$. Therefore, $x \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ implies $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$. The converse is obvious and $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_0. ■

Definition 3.18. A filterbase $\mathcal{F}$ is called $\mathcal{G}$-semiconvergent to a point x in X, if for any $U \in \mathcal{G}SO(X, x)$, there exists $B \in \mathcal{F}$ such that B is a subset of U.

Lemma 3.19. Let $(X, \tau, \mathcal{G})$ be a grill topological space and let x and y be any two points in X such that every net in X $\mathcal{G}$-semiconverging to y $\mathcal{G}$-semiconverges to x. Then $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$.

Proof. Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n \in N}$ is a net in $s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Since $\{x_n\}_{n \in N}$ $\mathcal{G}$-semiconverges to y, then $\{x_n\}_{n \in N}$ $\mathcal{G}$-semiconverges to x and this implies that $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. ■

Theorem 3.20. For a grill topological space $(X, \tau, \mathcal{G})$, the following statements are equivalent:

(1) $(X, \tau, \mathcal{G})$ is a $\mathcal{G}$-semi-R_0 space;

(2) If $x, y \in X$, then $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ if and only if every net in X $\mathcal{G}$-semiconverging to y $\mathcal{G}$-semiconverges to x.

Proof. (1) $\Rightarrow$ (2): Let $x, y \in X$ such that $y \in s\mathrm{Cl}G(\{x\})$. Suppose that $\{x_\alpha\}_{\alpha\in N}$ be a net in X such that $\{x_\alpha\}_{\alpha\in N}$ $\mathcal{G}$-semiconverges to y. Since $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$, by Theorem 3.10 we have $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Therefore $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. This means that $\{x_\alpha\}_{\alpha\in N}$ $\mathcal{G}$-semiconverges to x. Conversely, let $x, y \in X$ such that every net in X $\mathcal{G}$-semiconverging to y $\mathcal{G}$-semiconverges to x. Then $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ by Lemma 3.3. By Theorem 3.10, we have $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Therefore $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$.
(2) $\Rightarrow$ (1): Assume that x and y are any two points of X such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \neq \varnothing$. Let $z \in s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha\in N}$ in $s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ such that $\{x_\alpha\}_{\alpha\in N}$ $\mathcal{G}$-semiconverges to z. Since $z \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, then $\{x_\alpha\}_{\alpha\in N}$ $\mathcal{G}$-semiconverges to y. It follows that $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$. By the same token we obtain $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Therefore $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ and by Theorem 3.10 $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_0. ■

4. On $\mathcal{G}$-semi-R_1 spaces

Definition 4.1. A grill topological space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}$-semi-R_1 if for x, y in X with $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, there exist disjoint $\mathcal{G}$- semiopen sets U and V such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset U$ and $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \subset V$.

Proposition 4.2. If $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_1, then it is $\mathcal{G}$-semi-R_0.

Proof. Let $U \in \mathcal{G}SO(X, x)$. If $y \notin U$, then since $x \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Hence there exists a $\mathcal{G}$-semiopen V_y such that $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \subset V_y$ and $x \notin V_y$, which implies $y \notin s\mathrm{Cl}_{\mathcal{G}}(\{x\})$. Thus $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset U$. Therefore $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_0. ■

Theorem 4.3. A grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_1 if and only if for $x, y \in X$, $s\mathrm{Ker}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Ker}_{\mathcal{G}}(\{y\})$, there exist disjoint $\mathcal{G}$-semiopen sets U and V such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset U$ and $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \subset V$.

Proof. It follows from Lemma 3.11. ■

Theorem 4.4. The following properties are equivalent:

(1) $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-T_2,

(2) $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_1 and $\mathcal{G}$-semi-T_1, and

(3) $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_1 and $\mathcal{G}$-semi-T_0.

Proof. (1) $\Rightarrow$ (2): Since $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-T_2, then it is $\mathcal{G}$-semi-T_1. If $x, y \in X$ such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, then $x \neq y$ and there exist disjoint $\mathcal{G}$-semiopen sets U and V such that $x \in U$ and $y \in V$ and $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = \{x\} \subset U$ and $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = \{y\} \subset V$. Hence $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_1.
(2) $\Rightarrow$ (3): Since $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-T_1, then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-T_0.
(3) $\Rightarrow$ (1): Since $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_1, and $\mathcal{G}$-semi-T_1, then $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_0 and $\mathcal{G}$-semi-T_0, which implies $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-T_1. Let $x, y \in X$ such that $x \neq y$. Since $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = \{x\} \neq \{y\} = s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, then there exist disjoint $\mathcal{G}$-semiopen sets U and V such that $x \in U$ and $y \in V$. Hence, $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-T_2. ■

Theorem 4.5. The following properties are equivalent:

(1) $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_1 ,

(2) for each $x, y \in X$ one of the following holds:

  (a) If U is $\mathcal{G}$-semiopen, then $x \in U$ if and only if $y \in U$.

  (b) there exist disjoint $\mathcal{G}$-semiopen sets U and V such that $x \in U$ and $y \in V$, and

(3) If $x, y \in X$ such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, then there exist $\mathcal{G}$-semiclosed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_2$, $y \in F_1$, $x \notin F_2$, and $X = F_1 \cup F_2$.

Proof. (1) $\Rightarrow$ (2): Let $x, y \in X$. Then $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ or $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. If $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ and U is $\mathcal{G}$-semiopen, then $x \in U$ implies $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset U$ and $y \in U$ implies $x \in s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \subset U$. Thus consider the case that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Then there exist disjoint $\mathcal{G}$-semiopen sets U and V such that $x \in U$ and $y \in V$ and $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = \{x\} \subset U$ and $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = \{y\} \subset V$. Hence $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_1.
(2) $\Rightarrow$ (3): Let $x, y \in X$ such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}G(y)$. Then $x \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ or $y \notin s\mathrm{Cl}_{\mathcal{G}}(\{x\})$, say $x \notin s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Then there exist a $\mathcal{G}$-semiopen set A such that $x \in A$ and $y \notin A$, which implies there exist disjoint $\mathcal{G}$-semiopen sets U and V such that $x \in U$ and $y \in V$. Then $F_1 = X\backslash V$ and $F_2 = X\backslash U$ are $\mathcal{G}$-semiclosed sets such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, $X = F_1 \cup F_2$.
(3) $\Rightarrow$ (1): Let U be $\mathcal{G}$-semiopen and let $x \in U$. Then $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset U$, for suppose not. Let $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \cap (X\backslash U)$. Then $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ and there exist $\mathcal{G}$-semiclosed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \in F_2$, $x \notin F_2$, $X = F_1 \cup F_2$. Then $y \in F_2\backslash F_1 = X\backslash F_1$, which is $\mathcal{G}$-semiopen, and $x \notin X\backslash F_1$, which is a contradiction. Hence, $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_0. Let $a, b \in X$ such that $s\mathrm{Cl}_{\mathcal{G}}(\{a\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{b\})$. Then there exist $\mathcal{G}$-semiclosed sets A_1 and A_2 such that $a \in A_1$, $b \notin A_1$, $a \notin A_2$, and $X = A_1 \cup A_2$. Thus $a \in A_1\backslash A_2$ and $b \in A_2\backslash A_1$, which are $\mathcal{G}$-semiopen, which implies $s\mathrm{Cl}_{\mathcal{G}}(\{a\}) \subset A_1\backslash A_2$ and $s\mathrm{Cl}_{\mathcal{G}}(\{b\}) \subset A_2\backslash A_1$. Hence, $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_1. ■

Theorem 4.6. A grill topological space $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-T_2 if and only if for $x, y \in X$ such that $x \neq y$, there exist $\mathcal{G}$-semiclosed sets F_1 and F_2 such that $x \in F_1$, $y \notin F_1$, $y \notin F_2$, $x \notin F_2$, and $X = F_1 \cup F_2$.

Proof. The straightforward proof is omitted. ■

Remark 4.7. If $\{x_\lambda\}_{\lambda\in A}$ is a net in $(X,\tau,\mathcal{G})$, $\mathcal{G}S\lim(\{x_\lambda\}_{\lambda\in A}) = \{x \in X : \{x_\lambda\}_{\lambda\in A} s-\mathcal{G}$-converges to x$\}$.

Theorem 4.8. The following properties are eqiuvalent:

(1) $(X,\tau,\mathcal{G})$ is $\mathcal{G}$-semi-R_1;

(2) for $x, y \in X$, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, whenever there exists a net $\{x_\lambda\}_{\lambda\in A}$ such that $x, y \in \mathcal{G}p\lim(\{x_\lambda\}_{\lambda\in A})$;

(3) $(X,\tau,\mathcal{G})$ is $\mathcal{G}$-semi-R_0, and for every $\mathcal{G}$-semiconvergent net $\{x_\lambda\}_{\lambda\in A}$ in X, $\mathcal{G}p\mathrm{lim}(\{x_\lambda\}_{\lambda\in A}) = s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ for some $x \in X$.

Proof. (1) $\Rightarrow$ (2): Let $x, y \in X$ such that there exists a net $\{x_\lambda\}_{\lambda\in A}$ in X such that $x, y \in \mathcal{G}p\,lim(\{x_\lambda\}_{\lambda\in A})$. Then (a) if U is $\mathcal{G}$-semiopen, then $x \in U$ if and only if $y \in U$ or (b) there exist disjoint $\mathcal{G}$-semiopen sets U and V such that $x \in U$ and $y \in V$. Since $x, y \in \mathcal{G}S\,lim(\{x_\lambda\}_{\lambda\in A})$, then (1) is satisfied, which implies $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Cl}_{\mathcal{G}}(\{y\})$.
(2) $\Rightarrow$ (3): Let $U \in \mathcal{G}SO(X,x)$. Let $y \notin U$. For each $n \in N$ let $x_n = x$. Then $\{x_n\}_{n\in N}$ $\mathcal{G}$-semiconverges to x and since $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, that $y \in A$ and $x \notin A$. Thus, $y \notin s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ and $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \subset U$. Hence $(X,\tau,\mathcal{G})$ is $\mathcal{G}$-semi-R_0. Let $\{x_\lambda\}_{\lambda\in A}$ be a $\mathcal{G}$-semiconvergent net in X. Let $x \in X$ such that $\{x_\lambda\}_{\lambda\in A}$ $\mathcal{G}$-semiconverges to x. If $y \in s\mathrm{Cl}_{\mathcal{G}}(\{x\})$, then $\{x_\lambda\}_{\lambda\in A}$ $\mathcal{G}$-semiconverges to y, which implies $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset \mathcal{G}S\,lim(\{x_\lambda\}_{\lambda\in A})$ and if $y \in \mathcal{G}plim(\{x_\lambda\}_{\lambda\in A})$, then $x, y \in \mathcal{G}plim(\{x_\lambda\}_{\lambda\in A})$, which implies $y \in s\mathrm{Cl}_{\mathcal{G}}(\{y\}) = s\mathrm{Cl}_{\mathcal{G}}(\{x\})$. Hence $\mathcal{G}S\,lim(\{x_\lambda\}_{\lambda\in A}) = s\mathrm{Cl}_{\mathcal{G}}(\{x\})$.
(3) $\Rightarrow$ (1): Assume that $(X,\tau,\mathcal{G})$ is not $\mathcal{G}$-semi-R_1. Then there exist $x, y \in X$ such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$ and every $\mathcal{G}$-semiopen set containing $s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ intersects every $\mathcal{G}$-semiopen set containing $s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Since $(X,\tau,\mathcal{G})$ is $\mathcal{G}$-semi-R_0, then every $\mathcal{G}$-semiopen set containing x contains $s\mathrm{Cl}_{\mathcal{G}}(\{x\})$ and every $\mathcal{G}$-semiopen set containing y contains $s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, which implies that every $\mathcal{G}$-semiopen set containing x intersects every $\mathcal{G}$-semiopen set containing y. Let $D_x = \{U \subset X : U \in \mathcal{G}SO(X,x)\}$. Let $\geq_x$ be the binary relation on D_x defined by $U_1 \geq_x U_2$ if and only if $U_1 \subset U_2$. Then, clearly $(D_x, \geq_x)$ is a directed set. Let $D_y = \{U \subset X : U \in \mathcal{G}SO(X,y)\}$ and let $\geq_y$ be the binary relation on D_y defined by $U_1 \geq_y U_2$ if and only if $U_1 \subset U_2$. Then, $(D_y, \geq_y)$ is also a directed set. Let $D = \{(U_1, U_2) : U_1 \in D_x \text{ and } U_2 \in D_y\}$ and let $\geq$ be the binary relation on D defined by $(U_1, U_2) \geq (V_1, V_2)$ if and only if $U_1 \geq_x V_1$ and $U_2 \geq_y V_2$. Then, $(D, \geq)$ is a directed set. For each $(U_1, U_2) \in D$, let $x_{(U_1,U_2)} \in (U_1, U_2)$. Then $\{x_{(U_1,U_2)}\}_{(U_1,U_2)\in D}$ is a net in X that $\mathcal{G}$-semiconverges to both x and y. Thus, there exists $z \in X$ such that $\mathcal{G}p\lim(\{x_{(U_1,U_2)}\}_{(U_1,U_2)\in D}) = s\mathrm{Cl}_{\mathcal{G}}(\{z\})$, which implies $x, y \in s\mathrm{Cl}_{\mathcal{G}}(\{z\})$. Since $\{s\mathrm{Cl}_{\mathcal{G}}(\{w\}) : w \in X\}$ is a decomposition of X, then $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) = s\mathrm{Cl}_{\mathcal{G}}(\{z\}) = s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, which is a contradiction. Hence $(X,\tau,\mathcal{G})$ is $\mathcal{G}$-semi-R_1. ■

Theorem 4.9. A grill topological space $(X,\tau,\mathcal{G})$ is $\mathcal{G}$-semi-T_2 if and only if every $\mathcal{G}$-semiconvergent net in X $\mathcal{G}$-semiconverges to a unique point.

Proof. The proof follows from Theorems 4.4 and 4.8. ■

Theorem 4.10. The following properties are eqiuvalent:

(1) $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_1,

(2) for each pair $x, y \in X$, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, there exists a $\mathcal{G}$-semiopen, $\mathcal{G}$-semiclosed set V such and $y \notin V$;

(3) for each pair $x, y \in X$, $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$, there exists a G-semicontinuous function $f : (X, \tau, \mathcal{G}) \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.

Proof. (1) $\Rightarrow$ (2): Let $x, y \in X$ such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Then there exist disjoint $\mathcal{G}$-semiopen sets U and W such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset U$ and $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \subset W$ and $V = s\mathrm{Cl}_{\mathcal{G}}(U)$ is $\mathcal{G}$-semiopen, $\mathcal{G}$-semiclosed such that $x \in V$ and $y \notin V$.
(2) $\Rightarrow$ (3): Let $x, y \in X$ such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Let V be $\mathcal{G}$-semiopen, $\mathcal{G}$-semiclosed set of X such that $x \in V$ and $y \notin V$. Thus, the function $f : (X, \tau, \mathcal{G}) \to [0, 1]$ defined by $f(z) = 0$ if $z \in V$ and $f(z) = 1$ if $z \notin V$ satisfies the desired properties.
(3) $\Rightarrow$ (1): Let $x, y \in X$ such that $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \neq s\mathrm{Cl}_{\mathcal{G}}(\{y\})$. Let $f : (X, \tau, \mathcal{G}) \to [0, 1]$ such that f is $\mathcal{G}$-semicontinuous, $f(x) = 0$ and $f(y) = 1$. Then $U = f^{-1}([0, 0.5))$ and $V = f^{-1}((0.5, 1])$ are disjoint such that $\mathcal{G}$-semiopen, $\mathcal{G}$-semiclosed set of X and $s\mathrm{Cl}_{\mathcal{G}}(\{x\}) \subset U$ and $s\mathrm{Cl}_{\mathcal{G}}(\{y\}) \subset V$. ■

Theorem 4.11. The following properties are eqiuvalent.

Proof.

(1) $(X, \tau, \mathcal{G})$ is $\mathcal{G}$-semi-R_1,

(2) for each pair $x, y \in X, x \neq y$, there exists a $\mathcal{G}$-semiopen, $\mathcal{G}$-semiclosed set V such and $y \notin V$;

(3) for each pair $x, y \in X$, $x \neq y$, there exists a $\mathcal{G}$-semicontinuous function $f : (X, \tau, \mathcal{G}) \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$. ■

Theorem 4.12.

(1) A grill topological space $(X, \tau, \mathcal{P}(X)\backslash\{\emptyset\})$ is $\mathcal{G}$-semi-R_0 (resp. $\mathcal{G}$-semi-R_1) if and only if it is pre-R_0 (resp. pre-R_1).

(2) A grill topological space (X, τ, X) is $\mathcal{G}$-semi-R_0 (resp. $\mathcal{G}$-semi-R_1) if and only if it is R_0 (resp. R_1).

Proof. The proof follows from Corollaries 3.3 and 3.4 of [13]. ■

References

[1] N. A. Shanin. On separation in topological spaces. *Dokl. Akad. Nauk. SSSR*, 38:110–113, 1943.

[2] A. S. Davis. Indexed systems of neighborhoods for general topological spaces. *Amer. Math. Monthly*, 68:886–893, 1961.

[3] K. K. Dube. A note on R_0 topological spaces. *Mat. Vesnik*, 11:203–208, 1974.

[4] D. W. Hall, S. K. Murphy, and J. Rozycki. On spaces which are essentially T_1. *J. Austr. Math. Soc.*, 12:451–455, 1971.

[5] S. A. Naimpally. On R_0-topological spaces. *Ann. Univ. Sci. Budapest Eotvos Sect. Math.*, 10:53–54, 1967.

[6] G. Choquet. Sur les notions de filter et grill. *Computes Rendus Acad. Sci. Paris*, 224:171–173, 1947.

[7] K. C.Chattopadhyay, O. Njastad, and W. J. Thron. Merotopic spaces and extensions of closure spaces. *Can. J. Math.*, 35(4):613–629, 1983.

[8] K. C. Chattopadhyay and W. J. Thron. Extensions of closure spaces. *Can. J. Math.*, 29(6):1277–1286, 1977.

[9] W. J. Thron. Proximity structure and grills. *Math. Ann.*, 206:35–62, 1973.

[10] B. Roy and M. N. Mukherjee. On a typical topology induced by a grill. *Soochow J. Math.*, 33(4):771–786, 2007.

[11] D. Mondal and M. N. Mukherjee. On a class of sets via grill: A decomposition of continuity. *An. St. Univ. Ovidius Constanta*, 20(1):307–316, 2012.

[12] M. Baskar and N. Rajesh. New separation axioms via $\mathcal{G}$-semiopen sets (submitted).

[13] A. Al-Omari and T. Noiri. Decomposition of continuity via grills. *Jordan J. Math and Stat.*, 4(1):33–46, 2011.