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A note on weighted Boole polynomials

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Abstract

The Boole polynomials play an important role in the area of number theory, algebra and umbral calculus. In this paper, we investigate some interesting properties of weighted Boole polynomials and consider Witt-type formula for the weighted Boole numbers and polynomials. Finally, we derive some new interesting identities and properties of those polynomials from the Witt-type formula which are related to the weighted Euler polynomials.

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1. Introduction

Let p be an odd prime number. $\mathbb{Z}_p$, $\mathcal{Q}_p$ and $\mathcal{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathcal{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ (see [12])}. \tag{1.1}$$

Let $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \text{ (see [3, 7-12])}. \tag{1.2}$$

It is well known that the *Euler polynomials of order* $k (\in \mathbb{N})$ are given by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}, \text{ (see [1-4, 7, 14])}. \tag{1.3}$$

When $k = 1$, $E_n(x) = E_n^{(1)}(x)$ are called the *ordinary Euler polynomials*, and in particular, if $x = 0$, $E_n = E_n(0)$ are called the *Euler numbers*.

The Changhee polynomials are defined by D. S. Kim et al which is given by the generating function to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}, \text{ (see [6])}. \tag{1.4}$$

When $x = 0$, $Ch_n = Ch_n(0)$ are called the Changhee numbers.

The *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l)x^l, \ (n \geq 0),$$

and the *Stirling numbers of the second kind* is defined by

$$(e^t - 1)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!}, \text{ (see [6, 8])}.$$

The *Boole polynomials* are defined by the generating function to be

$$\sum_{n=0}^{\infty} 2Bl_n(x|\lambda)\frac{t^n}{n!} = \frac{1}{\left(1+(1+t)^{\lambda}\right)}(1+t)^x, \text{ (see [5, 16, 17])}.$$

In viewpoint of the generalization of the Boole polynomials, we consider the *weighted Boole polynomials of order* $k (\in \mathbb{N})$ are defined by the generating function to be

$$2^k \sum_{n=0}^{\infty} Bl_{n,q^{\alpha}}^{(k)}(x|\lambda)\frac{t^n}{n!} = \left(\frac{2}{1+(1+q^{\alpha}t)^{\lambda}}\right)^k (1+q^{\alpha}t)^x. \tag{1.5}$$

When $k = 1$, $Bl_{n,q^{\alpha}}(x|\lambda) = Bl_{n,q^{\alpha}}^{(1)}(x|\lambda)$ are called the *weighted Boole polynomials*.

In this paper, we investigate some properties of weighted Boole polynomials and consider Witt-type formulas the weighted Boole numbers and polynomials. Finally, we derive some new identities of those polynomials from the Witt-type formulas which are related to weighted Euler polynomials.

2. The weighted Boole polynomials

In this section, we assume that $t \in \mathcal{C}_p$ with $|q^\alpha t|_p < p^{-\frac{1}{p-1}}$ and $\lambda \in \mathbb{Z}_p$. Let us take $f(x) = (1+q^\alpha t)^{\lambda x}$. From (1.2), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} (1+q^\alpha t)^{x+\lambda y} d\mu_{-1}(y) &= \frac{2}{1+(1+q^\alpha t)^\lambda}(1+q^\alpha t)^x \\ &= \sum_{n=0}^{\infty} 2Bl_{n,q^\alpha}(x|\lambda)\frac{t^n}{n!}.\end{aligned} \tag{2.1}$$

It is easy to show that

$$\int_{\mathbb{Z}_p} (1+q^\alpha t)^{x+\lambda y} d\mu_{-1}(y) = \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} q^{\alpha n}(x+\lambda y)_n d\mu_{-1}(y)\frac{t^n}{n!} \tag{2.2}$$

Therefore, by (2.1) and (2.2), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$\int_{\mathbb{Z}_p} q^{\alpha n}(x+\lambda y)_n d\mu_{-1}(y) = 2Bl_{n,q^\alpha}(x|\lambda).$$

By replacing t by $\frac{1}{q^\alpha}(e^t - 1)$ in (1.5), we get

$$\begin{aligned}\sum_{n=0}^{\infty} 2Bl_{n,q^\alpha}(x|\lambda)\frac{\left(\frac{1}{q^\alpha}(e^t-1)\right)^n}{n!} &= \frac{2}{e^{\lambda t}+1}e^{xt} \\ &= \sum_{n=0}^{\infty} E_n\left(\frac{x}{\lambda}\right)\lambda^n\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} 2Bl_{n,q^\alpha}(x|\lambda)\frac{1}{n!}\left(\frac{1}{q^\alpha}(e^t-1)\right)^n &= \sum_{n=0}^{\infty} 2Bl_{n,q^\alpha}(x|\lambda)\frac{1}{q^{\alpha n}n!}n!\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(2\sum_{n=0}^{m}\frac{1}{q^{\alpha n}}Bl_{n,q}(x|\lambda)S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.4}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.2. For $m \geq 0$, we have

$$\sum_{n=0}^{m} \frac{Bl_{n,q^{\alpha}}(x|\lambda) S_2(m,n)}{q^{\alpha n}} = \frac{1}{2} \lambda^m E_m \left(\frac{x}{\lambda} \right).$$

From Theorem 2.1, we note that

$$\begin{aligned} 2Bl_{n,q^{\alpha}}(x|\lambda) &= \sum_{l=0}^{n} q^{\alpha n} S_1(n,l) \int_{\mathbb{Z}_p} (x + y\lambda)^l d_{\mu_{-1}}(y) \\ &= \sum_{l=0}^{\infty} q^{\alpha n} S_1(n,l) \lambda^l E_l \left(\frac{x}{\lambda} \right). \end{aligned} \tag{2.5}$$

Thus, by (2.5), we obtain the following corollary.

Corollary 2.3. For $n \geq 0$, we have

$$Bl_{n,q^{\alpha}}(x|\lambda) = \frac{1}{2} \sum_{l=0}^{\infty} q^{\alpha n} S_1(n,l) \lambda^l E_l \left(\frac{x}{\lambda} \right).$$

Let us consider the Boole polynomials of order $k (\in \mathbb{N})$ as follows:

$$2^k Bl_{n,q^{\alpha}}^{(k)}(x|\lambda) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\alpha n} \, (\lambda x_1 + \cdots + \lambda x_k + x)_n \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{2.6}$$

Thus, by (2.6), we get

$$2^k Bl_{n,q^{\alpha}}^{(k)}(x|\lambda) = \sum_{l=0}^{n} q^{\alpha n} S_1(n,l) \lambda^l E_l^{(k)} \left(\frac{x}{\lambda} \right).$$

From (2.6), we can derive the generating function of $Bl_{n,q^{\alpha}}^{(k)}(x)$ as follows:

$$\begin{aligned} &2^k \sum_{n=0}^{\infty} Bl_{n,q^{\alpha}}^{(k)}(x|\lambda) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + q^{\alpha} t)^{\lambda x_1 + \cdots + \lambda x_k + x} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \\ &= \left(\frac{2}{1 + (1 + q^{\alpha} t)^{\lambda}} \right)^k (1 + q^{\alpha} t)^x. \end{aligned} \tag{2.7}$$

By replacing t by $\frac{1}{q^{\alpha}} \left(e^t - 1 \right)$ in (2.7), we get

$$\begin{aligned} 2^k \sum_{n=0}^{\infty} Bl_{n,q^{\alpha}}^{(k)}(x|\lambda) \frac{1}{n!} \left(\frac{1}{q^{\alpha}} \left(e^t - 1 \right) \right)^n &= \left(\frac{2}{e^{\lambda t} + 1} \right)^k e^{xt} \\ &= \sum_{m=0}^{\infty} E_m^{(k)} \left(\frac{x}{\lambda} \right) \lambda^m \frac{t^m}{m!}, \end{aligned} \tag{2.8}$$

and

$$2^k \sum_{n=0}^{\infty} Bl_{n,q^\alpha}^{(k)}(x|\lambda) \frac{\left(\frac{1}{q^\alpha}\left(e^t-1\right)\right)^n}{n!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} \frac{2^k}{q^{\alpha n}} Bl_{n,q^\alpha}^{(k)}(x|\lambda) S_2(m,n)\right)\frac{t^m}{m!}. \quad (2.9)$$

Therefore, by (2.8) and (2.9), we obtain the following theorem.

Theorem 2.4. For $m \geq 0$, we have

$$\sum_{n=0}^{m} \frac{Bl_{n,q^\alpha}^{(k)}(x|\lambda) S_2(m,n)}{q^{\alpha n}} = \frac{\lambda^m}{2^k} E_m^{(k)}\left(\frac{x}{\lambda}\right)$$

and

$$Bl_{n,q^\alpha}^{(k)}(x|\lambda) = \frac{q^{\alpha n}}{2^k}\sum_{l=0}^{n} S_1(n,l)\lambda^l E_l^{(k)}\left(\frac{x}{\lambda}\right).$$

For $n \geq 0$, the rising factorial sequence is defined by

$$\begin{aligned} x^{(n)} &= x(x+1)\cdots(x+n-1) = (-1)^n(-x)_n \\ &= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix} x^l, \end{aligned}$$

where $\begin{bmatrix} n \\ l \end{bmatrix} = (-1)^{n+l} S_1(n,l)$.

Now, we define the Boole polynomials of the second kind as follows:

$$\widehat{Bl}_{n,q^\alpha}(x|\lambda) = \frac{1}{2}\int_{\mathbb{Z}_p} (-\lambda y + x)_n d\mu_{-1}(y), \ (n \geq 0). \quad (2.10)$$

Then, by (2.10), we get

$$\begin{aligned} \widehat{Bl}_{n,q^\alpha}(x|\lambda) &= \frac{1}{2}\sum_{l=0}^{n} S_1(n,l)(-1)^l\lambda^l \int_{\mathbb{Z}_p}\left(-\frac{x}{\lambda}+y\right)^l d\mu_{-1}(y) \\ &= \frac{1}{2}\sum_{l=0}^{n} S_1(n,l)(-1)^l\lambda^l E_l\left(\frac{x}{\lambda}\right). \end{aligned}$$

When $x = 0$, $\widehat{Bl}_{n,q^\alpha}(\lambda) = \widehat{Bl}_{n,q^\alpha}(0|\lambda)$ are called the *weighted Boole numbers of the second kind*.

By (2.10), the generating function of $\widehat{Bl}_{n,q^\alpha}(x|\lambda)$ is given by

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{Bl}_{n,q^\alpha}(x|\lambda)\frac{t^n}{n!} &= \frac{1}{2}\int_{\mathbb{Z}_p} (1+q^\alpha t)^{-\lambda y + x} d\mu_{-1}(y) \\ &= \frac{(1+q^\alpha t)^\lambda}{1+(1+q^\alpha t)^\lambda}(1+q^\alpha t)^x. \end{aligned}$$

By replacing t by $\frac{1}{q^\alpha}(e^t-1)$, we have

$$\sum_{n=0}^{\infty}\widehat{Bl}_{n,q^\alpha}(x|\lambda)\frac{1}{n!}\left(\frac{1}{q^\alpha}\left(e^t-1\right)\right)^n=\sum_{m=0}^{\infty}\frac{\lambda^m}{2}E_m\left(\frac{\lambda+x}{\lambda}\right)\frac{t^m}{m!}, \tag{2.11}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty}\widehat{Bl}_{n,q^\alpha}(x|\lambda)\frac{1}{n!}\left(\frac{1}{q^\alpha}\left(e^t-1\right)\right)^n\\ =&\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\frac{1}{q^{\alpha n}}\widehat{Bl}_{n,q^\alpha}(x|\lambda)S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.12}$$

Therefore, by (2.11) and (2.12), we obtain the following theorem.

Theorem 2.5. For $m\geq 0$, we have

$$\frac{\lambda^m}{2}E_{m,q}\left(1+\frac{x}{\lambda}\right)=\sum_{n=0}^{m}\frac{\widehat{Bl}_{n,q^\alpha}(x|\lambda)S_2(m,n)}{q^{\alpha n}},$$

and

$$\widehat{Bl}_{m,q^\alpha}(x|\lambda)=\frac{1}{2}\sum_{l=0}^{m}S_1(m,l)(-\lambda)^lE_l\left(-\frac{x}{\lambda}\right).$$

For $k\in\mathbb{N}$, let us consider the *weighted Boole polynomials of the second kind with order k* as follows:

$$\begin{aligned}&\widehat{Bl}_{n,q^\alpha}(x|\lambda)\\ =&\frac{1}{2^k}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}q^{\alpha n}(-\lambda x_1-\cdots-\lambda x_k+x)_n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).\end{aligned} \tag{2.13}$$

Thus, by (2.13), we get

$$2^k\widehat{Bl}_{n,q^\alpha}(x|\lambda)=\sum_{l=0}^{n}q^{\alpha n}S_1(n,l)(-\lambda)^lE_l^{(k)}\left(-\frac{x}{\lambda}\right). \tag{2.14}$$

By (2.13), the generating function of $\widehat{Bl}^{(k)}_{n,q^\alpha}(x|\lambda)$ is given by

$$\begin{aligned}&\sum_{n=0}^{\infty}\widehat{Bl}^{(k)}_{n,q^\alpha}(x|\lambda)\frac{t^n}{n!}\\ =&\frac{1}{2^k}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+q^\alpha t)^{-\lambda x_1-\cdots-\lambda x_k+x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\\ =&\left(\frac{(1+q^\alpha t)^\lambda}{1+(1+q^\alpha t)^\lambda}\right)^k(1+q^\alpha t)^x.\end{aligned} \tag{2.15}$$

By replacing t by $\frac{1}{q^\alpha}\left(e^t-1\right)$, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\widehat{Bl}_{n,q^\alpha}^{(k)}(x|\lambda)\frac{1}{n!}\left(\frac{1}{q^\alpha}\left(e^t-1\right)\right)^n &= \frac{1}{2^k}\left(\frac{2}{e^{\lambda t}+1}\right)^k e^{(\lambda k+x)t} \\ &= \sum_{m=0}^{\infty}\frac{\lambda^m}{2^k}E_m^{(k)}\left(k+\frac{x}{\lambda}\right)\frac{t^m}{m!}\end{aligned} \tag{2.16}$$

and

$$\sum_{n=0}^{\infty}\widehat{Bl}_{n}^{(k)}(x|\lambda)\frac{1}{n!}\left(\frac{1}{q^\alpha}\left(e^t-1\right)\right)^n = \sum_{m=0}^{\infty}\Big(\sum_{n=0}^{m}q^{\alpha n}\widehat{Bl}_{n,q^\alpha}^{(k)}(x|\lambda)S_2(m,n)\Big)\frac{t^m}{m!}. \tag{2.17}$$

Therefore, by (2.16) and (2.17), we obtain the following theorem.

Theorem 2.6. For $m \geq 0$, we have

$$\widehat{Bl}_{n,q^\alpha}^{(k)}(x|\lambda) = \frac{1}{2^k}\sum_{l=0}^{m}S_1(m,l)(-\lambda)^l E_l^{(k)}\left(-\frac{x}{\lambda}\right)$$

and

$$E_m^{(k)}\left(k+\frac{x}{\lambda}\right) = \frac{2^k}{\lambda^m}\sum_{n=0}^{m}q^{\alpha n}\widehat{Bl}_{n,q^\alpha}^{(k)}(x|\lambda)S_2(m,n).$$

Now, we observe that

$$\begin{aligned}(-1)^n\frac{2Bl_{n,q^\alpha}(x|\lambda)}{n!} &= (-1)^n\int_{\mathbb{Z}_p}\binom{x+y\lambda}{n}d\mu_{-1}(y) \\ &= \int_{\mathbb{Z}_p}\binom{-y\lambda-x+n-1}{n}d\mu_{-1}(y) \\ &= \sum_{m=0}^{n}\binom{n-1}{n-m}\int_{\mathbb{Z}_p}\binom{-y\lambda-x}{m}d\mu_{-1}(y) \\ &= \sum_{m=1}^{n}\frac{\binom{n-1}{m-1}}{m!}m!\int_{\mathbb{Z}_p}\binom{-y\lambda-x}{m}d\mu_{-1}(y) \\ &= \sum_{m=1}^{n}\binom{n-1}{m-1}\frac{2\widehat{Bl}_{m,q^\alpha}^{(k)}(-x|\lambda)}{m!},\end{aligned} \tag{2.18}$$

and

$$(-1)^n \frac{2\widehat{Bl}_{n,q^\alpha}(x|\lambda)}{n!} = \sum_{m=0}^{n} \binom{n-1}{m-1} \int_{\mathbb{Z}_p} \binom{-x+y\lambda}{m} d\mu_{-1}(y) = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{2Bl_{m,q^\alpha}^{(k)}(-x|\lambda)}{m!}. \tag{2.19}$$

Therefore, by (2.18) and (2.19), We obtain the following theorem.

Theorem 2.7. For $n \geq 1$, we have

$$(-1)^n \frac{Bl_{n,q^\alpha}(x|\lambda)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{Bl}_{m,q^\alpha}^{(k)}(-x|\lambda)}{m!}$$

and

$$(-1)^n \frac{\widehat{Bl}_{n,q^\alpha}(x|\lambda)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{Bl_{m,q^\alpha}^{(k)}(-x|\lambda)}{m!}$$

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