Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 4 (2015), pp. 2031-2037

http://www.ripublication.com

Lateral and Right Ideals in Ternary Γ-Semigroups

Jyothi. V(1), Dr. Sarala. Y(2), Dr. Madhusudhana Rao. D(3), Dr. Anjaneyulu. A(4)

[1]Research Scholar, Department of Mathematics, K. L. University, A. P, India.
[2]Faculty of Mathematics, K. L. University, A. P, India.
[3]Faculty of Mathematics, VSR & NVR College, Tenali, A. P, India.
[4]Retired HOD, Department of Mathematics, VSR & NVR College, Tenali, A. P, India.
saralayella1970@gmail. com

Abstract

The main goal of this paper is to introduce the notion of a ternary Γ-semigroup; lateral ideal, maximal lateral ideal, lateral ideal generated by a subset, principal lateral ideal and lateral simple ternary Γ-semigroup are introduced and characterized each of them. In the next section we introduce right ideals in ternary Γ-semigroups and characterised.

Keywords: Ternary Γ-semigroups; Lateral ideal; Principal lateral ideal, right ideal.

Introduction:

The concept of semigroup is so simple and natural that it is hard to say when it first appeared. The theory of ternary algebraic systems was introduced by LEHMER [5] in 1932, but earlier such structures was studied by KASNER [4] who give the idea of *n*-ary algebras. LEHMAR

[5] investigated certain algebraic systems called triplexes which turnout to be commutative ternary groups. The notion of ternary semigroups was known to BANACH who is credited with example of a ternary semigroup which can not reduce to a semigroup. A. Anjaneyulu [1] studies about structure and ideal theory of semigroups. Sen. M. K, Sahan. K [9] Studied about Γ-semigroups. Further Y. Sarala [8] study about the ideals in ternary semigroups. In this paper mainly we extended the results to ternary Γ-semigroups.

Definition 2. 1:

Let T and Γ be two non-empty sets. Then T is said to be a ternary

Γ-semigroup if there exist a mapping from T X Γ X T X Γ X T to T which maps $(x_1, \alpha, x_2, \beta, x_3) \rightarrow [x_1\,\alpha\,x_2\,\beta\,x_3]$ satisfying the condition $[[x_1\,\alpha\,x_2\,\beta\,x_3]]\gamma\,x_4\,\delta\,x_5] = [x_1\,\alpha\,[x_2\,\beta\,x_3\,\gamma\,x_4]\,\delta\,x_5] = [x_1\,\alpha\,x_2\,\beta[x_3\,\gamma\,x_4\,\delta\,x_5]]$ $\forall\, x_i \in$ T; $1 \leq i \leq 5$ and $\alpha, \beta, \gamma, \delta \in$ Τ.

Example 2. 2:
T = {0, ±i} and T = Γ then T is ternary Γ-semigroup under the complex ternary operation.

Definition 2. 3:
A non-empty subset A of a ternary Γ-semigroup T is said to be a lateral ideal of T if b, c ∈ T; a ∈ A implies b$\alpha a\beta c \in$ A; $\alpha, \beta \in \Gamma$.

Note 2. 4:
A non-empty subset A of a ternary Γ-semigroup T is a lateral ideal of T if and only if TΓAΓT ⊆ A.

Theorem 2. 5:
The non-empty intersection of any two lateral ideals of a ternary Γ-semigroup T is a lateral ideal of T.

Proof:
Let A, B be two lateral ideals of T. Let $a \in A \cap B$ and $b, c \in$ T. $a \in A \cap B \Longrightarrow a \in A$ and $a \in B$. $a \in A$; $b, c \in$ T; A is a lateral ideal of T ⟹ b$\alpha a\beta c \in$ A; $\alpha, \beta \in \Gamma$. $a \in B$; $b, c \in$ T; B is a lateral ideal of T ⟹ b$\alpha a\beta c \in$ A; $\alpha, \beta \in \Gamma$. b$\alpha a\beta c \in$ A; b$\alpha a\beta c \in$ B ⟹b$\alpha a\beta c \in A \cap B$. Therefore $A \cap B$ is a lateral ideal of T.

Theorem 2. 6:
The non-empty intersection of any family of lateral ideals of a ternary Γ-semigroup T is a lateral ideal of T.

Proof:
Let $\{A_j\}_{j\in\Delta}$ be a family of lateral ideals of T and let A = $\cap_{j\in\Delta} A_j$. Let $a \in A$; $b, c \in$ T. Now $a \in A$; $a \in \cap_{j\in\Delta} A_j \Longrightarrow a \in A_j$ for each j ∈ Δ. $a \in A_j$; $b, c \in$ T; A_j is a lateral ideal of T implies b$\alpha a\beta c \in A_j$; $\alpha, \beta \in \Gamma \Longrightarrow b\alpha a\beta c \in A$. Therefore A is lateral ideal of T.

Theorem 2. 7:
The union of any two lateral ideals of a ternary Γ-semigroup T is a lateral ideal of T.

***Proof*:**
Let A_1, A_2 be two lateral ideals of a ternary Γ-semigroup T. Let A = $A_1 \cup A_2$. Clearly A is a nonempty subset of T. Let $a \in A$; $b, c \in$ T. $a \in A \Longrightarrow a \in A_1 \cup A_2 \Longrightarrow a \in A_1$ or $a \in A_2$. Suppose $a \in A_1$; $a \in A_1$, $b, c \in$ T; $\alpha, \beta \in \Gamma$, A_1 is a lateral ideal of $T \Longrightarrow b\alpha a\beta c \in A_1 \subseteq A_1 \cup A_2$ = A implies $b\alpha a\beta c \in A$. Suppose $a \in A_2$; $a \in A_2$, $b, c \in$ T and $\alpha, \beta \in \Gamma$;

A_2 is a lateral ideal of T implies $b\alpha a\beta c \in A_2 \subseteq A_1 \cup A_2 = A \Longrightarrow b\alpha a\beta c \in A$. Therefore $a \in A$; $b, c \in T$; $\alpha, \beta \in \Gamma \Longrightarrow b\alpha a\beta c \in A$ and hence A is lateral ideal of T.

Theorem 2. 8:
The union of any family of lateral ideals of a ternary Γ-semigroup T is lateral ideal of T.

Proof:
Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of lateral ideals of a ternary Γ-semigroup T. Let $A = \cup_{\alpha\in\Delta} A_\alpha$. Clearly A is a nonempty subset of T. Let $a \in A$; $b, c \in T$, $a \in A \Longrightarrow a \in \cup_{\alpha\in\Delta} A_\alpha \Longrightarrow a \in A_\alpha$ for some $\alpha \in \Delta$. $a \in A_\alpha$; $b, c \in T$; A_α is a lateral ideal of $T \Longrightarrow b\alpha a\beta c \in A_\alpha \subseteq \cup_{\alpha\in\Delta} A_\alpha = A$ implies $b\alpha a\beta c \in A$. Therefore A is lateral ideal of T.

We know introduce a maximal lateral ideal and lateral ideal generated by a subset of a ternary Γ-semigroup.

Definition 2. 9:
An ideal A of a ternary Γ-semigroup T is said to be a maximal lateral ideal provided A is a proper lateral ideal of T and is not properly contained in any proper lateral ideal of T.

Definition 2. 10:
Let T be a ternary Γ-semigroup and A be non-empty subset of T. The smallest lateral ideal of T containing A is called lateral ideal of T generated by A.

Theorem 2. 11:
The lateral ideal of a ternary Γ-semigroup T generated by a non-empty subset A is the intersection of all lateral ideals of T containing A.

Proof:
Let Δ be the set of all lateral ideals of T containing A. Since T itself is a lateral ideal of T containing A, $T \in \Delta$. So $\Delta \neq \emptyset$. Let $S^* = \cap_{S\in\Delta} S$. Since $A \subseteq S$ for all $S \in \Delta$, $A \subseteq S^*$. By theorem 2. 6; S^* is a lateral ideal of T. Let K be a lateral ideal of T containing A, Clearly $A \subseteq K$ and K is a lateral ideal of T. Therefore $K \in \Delta \Longrightarrow S^* \subseteq K$ and hence $S^*$is the lateral ideal of T generated by A. We now introduce a principal lateral ideal of a ternary Γ-semigroup and characterize principal lateral ideal.

Definition 2. 12:
A lateral ideal A of a ternary Γ-semigroup T is said to be the principal lateral ideal generated by a if A is a lateral ideal generated by $\{a\}$ for some $a \in T$. It is denoted by M (a) or $<a>_m$.

Theorem 2. 13:
If T is a ternary Γ-semigroup and $a \in T$ then $M(a) = a \cup T\Gamma a\Gamma T \cup T\Gamma T\,\Gamma a\, T\Gamma T$.

Proof:
Let s, $t \in$ T; $r \in a \cup$T$\Gamma a\Gamma$T$\cup$TΓT $\Gamma a\Gamma$TΓT; $r \in a \cup$T$\Gamma a\Gamma$T$\cup$TΓT $\Gamma a\Gamma$TT $\Rightarrow r$=a or r=$u\alpha a\beta v$ or $r = u\alpha v\beta a\gamma p\delta q$ for some u, v, p, q∈T. $\alpha, \beta, \gamma, \delta \in$ Γ. If r= a then $s\alpha r\beta t = s\ \alpha a\beta t \in$ T Γa ΓT ⊆ a∪ T Γa ΓT∪ TΓT ΓaΓTΓT. If $r = u\alpha a\beta v$ then $s\alpha_1 r\beta_1 t$ $= s\alpha_1(u\alpha a\beta v)\ \beta_1 t = s\alpha_1 u\alpha a\beta v\beta_1 t \in$ T Γa ΓT; $\alpha_1, \beta_1 \in$ Γ. If $r = u\alpha v\beta a\gamma p\delta q$ then $s\alpha_1 r\beta_1 t = s\alpha_1(u\alpha v\beta arp\delta q)\beta_1 t = (s\alpha_1 u\alpha v\beta a)\gamma(p\delta q\beta_1 t) \in$ TΓaΓT ⊆ a ∪TΓaΓT∪TΓTΓaΓTΓT. Therefore $s\alpha_1 r\beta_1 t \in a$ ∪TΓaΓT∪TΓT ΓaΓTΓT and hence a ∪TΓaΓT∪TΓTΓaΓTΓT is a lateral ideal of T. Let M be a lateral ideal of T containing a. Let $r \in a$ ∪TΓaΓT∪TΓT ΓaΓTΓT then r=a or r=$u\alpha a\beta v$ or $r = u\alpha v\beta a\gamma p\delta q$ for some $u, v, p, q \in$ T, $\alpha, \beta, \gamma, \delta \in$ Γ. If r =a then r=a ∈ M. If r=$u\alpha a\beta v$ then r=$u\alpha a\beta v$ ∈ M. If $r = u\alpha v\beta a\gamma p\delta q$ then $r = u\alpha v\beta a\gamma p\delta q$ ∈ M. Therefore a ∪TΓaΓT∪ TΓT ΓaΓTΓT ⊆ M and hence a ∪TΓaΓT∪TΓT ΓaΓTΓT is the smallest lateral ideal containing a. Therefore M(a) = a ∪TΓaΓT∪TΓT ΓaΓTΓT.

We now introduce a lateral simple ternary Γ-semigroup and characterize lateral simple ternary Γ-semigroup.

Definition 2. 14:
A ternary Γ-semigroup T is said to be a lateral simple ternary Γ-semigroup if T is its only lateral ideal

Theorem 2. 15:
A ternary Γ-semigroup T is a lateral simple ternary Γ-semigroup if and only if TΓaΓT= TΓT ΓaΓTΓT = T for all a ∈ T.

Proof:
Suppose that T is a lateral simple ternary Γ-semigroup and a ∈ T. Let s ∈ TΓaΓT implies s = v$\alpha a\beta w$ where v, w ∈ T and α, β∈ Γ. Now $u\alpha_1 s\beta_1 t = u\alpha_1(\mathrm{v}\alpha a\beta w)\ \beta_1 t$ $= u\alpha_1 \mathrm{v}\alpha a\beta w\beta_1\ t$ ∈ TΓTΓaΓTΓT= TΓaΓT ⟹TΓaΓT is a lateral ideal of T. Since T is a lateral simple ternary Γ-semigroup, TΓaΓT =T. Therefore TΓaΓT = T for all a ∈ T. Conversely suppose that TΓaΓT =T for all a ∈ T. Let M be a lateral ideal of T. Let m ∈ M. Then m ∈ T. By assumption TΓmΓT =T. Let t ∈ T then t ∈ TΓmΓT ⟹t $= u\alpha m\beta v$ for some u, v ∈ T and α, β ∈ Γ. m ∈ M; u, v ∈ T and M is a lateral ideal ⟹ u$\alpha m\beta v$ ∈ M ⟹ t ∈ M. Therefore T ⊆ M. Clearly M ⊆ T and hence M=T. Therefore T is only lateral ideal of T. Hence T is a lateral simple ternary Γ-semigroup.

Right Ideals in Ternary Γ-Semigroups

Definition 3. 1:
A non-empty subset A of a ternary Γ-semigroup T is a right ideal of T if b, c ∈ T; a ∈ A implies $a\alpha b\beta c$ ∈ A; α, β ∈ Γ.

Note 3. 2:
A non-empty subset A of a ternary Γ-semigroup T is a right ideal of T if and only if AΓTΓT ⊆ A.

Theorem 3. 3:
The non-empty intersection of any two right ideals of ternary Γ-semigroup T is a right ideal of T.

Proof:
Let A, B be two right ideals of T. $a \in A \cap B$ and $b, c \in$ T. $a \in A \cap B$ implies $a \in A$ and $a \in B$. $a \in A$; $b, c \in$ T, $\alpha, \beta \in \Gamma$, A is a right ideal of T $\Longrightarrow a\alpha b\beta c \in$ A. $a \in B$; $b, c \in$ T, $\alpha, \beta \in \Gamma$; B is a right ideal of T $\Longrightarrow a\alpha b\beta c \in$ B. $a\alpha b\beta c \in$ A; $a\alpha b\beta c \in$ B $\Longrightarrow a\alpha b\beta c \in A \cap B$. Therefore $A \cap B$ is a right ideal of T.

Theorem 3. 4:
The non-empty intersection of any family of right ideals of ternary Γ-semigroup T is a right ideal of T.

Proof:
Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of right ideals of T and let A $= \cap_{\alpha\in\Delta} A_\alpha$. Let $a \in$ A; $b, c \in$ T, $a \in$ A$\Longrightarrow a \in \cap_{\alpha\in\Delta} A_\alpha \Longrightarrow a \in A_\alpha$ for each $\alpha \in \Delta$. $a \in A_\alpha$; $b, c \in$ T; $\alpha_1, \beta_1 \in \Gamma$; A_α is a right ideal of T implies $a\alpha_1 b\beta_1 c \in A\alpha$ for all $\alpha\in\Delta \Longrightarrow a\alpha_1 b\beta_1 c \in \cap_{\alpha\in\Delta} A_\alpha \Longrightarrow a\alpha_1 b\beta_1 c \in A$. Therefore A is right ideal of T.

Theorem 3. 5:
The union of any two right ideals of a ternary Γ-semigroup T is a right ideal of T.

Proof:
Let A_1, A_2 be two right ideals of a ternary Γ-semigroup T. Let A $= A_1 \cup A_2$. Clearly A is a nonempty subset of T. Let $a \in A$; $b, c \in$ T. $a \in A \Longrightarrow a \in A_1 \cup A_2 \Longrightarrow a \in A_1$ or $a \in A_2$. Suppose $a \in A_1$; $a \in A_1$, $b, c \in$ T; $\alpha, \beta \in \Gamma$; A_1 is a right ideal of $T \Longrightarrow a\alpha b\beta c \in A_1 \subseteq A_1 \cup A_2$ = A implies $a\alpha b\beta c \in A$. Suppose $a \in A_2$; $a \in A_2$, $b, c \in$ T; $\alpha, \beta \in \Gamma$; A_2 is a right ideal of T implies $a\alpha b\beta c \in A_2 \subseteq A_1 \cup A_2$ = A $\Longrightarrow a\alpha b\beta c \in$ A. Therefore $a \in$ A; $b, c \in$ T and $\alpha, \beta \in \Gamma \Longrightarrow a\alpha b\beta c \in A$ and hence A is right ideal of T.

Theorem 3. 6:
The union of any family of right ideals of a ternary Γ-semigroup T is right ideal of T.

Proof:
Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of right ideals of a ternary Γ-semigroup T. Let A $= \cup_{\alpha\in\Delta} A_\alpha$. Clearly A is a nonempty subset of T. Let $a \in$ A; b, c $\in$ T. $a \in$ A$\Longrightarrow a \in \cup_{\alpha\in\Delta} A_\alpha \Longrightarrow a \in A_\alpha$ for some $\alpha \in \Delta$. $a \in A_\alpha$; $b, c \in$ T; $\alpha_1, \beta_1 \in \Gamma$; $A_\alpha$is a right ideal of T implies $a\alpha_1 b\beta_1 c \in A_\alpha \subseteq \cup_{\alpha\in\Delta} A_\alpha$=A implies $a\alpha_1 b\beta_1 c \in A$. Therefore A is right ideal of T.

We now introduce a maximal right ideal and right ideal generated by a subset of a ternary Γ-semigroup.

Definition 3. 7:
An ideal A of a ternary Γ-semigroup T is said to be a maximal right ideal provided A is a proper right ideal of T and is not properly contained in any proper right ideal of T.

Definition 3. 8:
Let T be a ternary Γ-semigroup and A be non-empty subset of T*.* The smallest right ideal of T containing A is called right ideal of T generated by A*.*

Theorem 3. 9:
The right ideal of a ternary Γ-semigroup T generated by a non-empty subset A is the intersection of all right ideals of T containing A*.*

Proof:
Let Δ be the set of all right ideals of T containing A. Since T itself is a right ideal of T containing A. T ∈ Δ. So $\Delta \neq \emptyset$. Let $S^* = \cap_{S\in\Delta} S$. Since A ⊆ S for all S ∈ Δ, A ⊆ S^*. By theorem 3. 4, S^* is a right ideal of T. Let K be a right ideal of T containing A. Clearly A ⊆ K and K is a right ideal of T. Therefore K ∈ Δ ⟹ S^* ⊆ K. Therefore $S^*$is the right ideal of T generated by A.

We now introduce a principal right ideal of a ternary Γ-semigroup and characterize principal right ideal.

Definition 3. 10:
A right ideal A of a ternary Γ-semigroup T is said to be a principal right ideal generated by a if A is a right ideal generated by $\{a\}$ for some a ∈ T. It is denoted by R (a) or $<a>_r$.

Theorem 3. 11:
If T is a ternary Γ-semigroup T and a ∈ T then R (a) = a∪aΓTΓT.

Proof:
Let s, t ∈ T; r ∈ a∪aΓTΓT; r ∈ a∪aΓTΓT ⟹ r=a or r= $a\alpha u\beta v$ for some u, v∈T, α, β ∈Γ. If r=a then $r\alpha_1 s\beta_1 t = a\alpha_1 s\beta_1 t$ ∈ a Γ T ΓT ⊆ a ∪ a Γ T ΓT. If $r = a\alpha u\beta v$ then $r\alpha_1 s\beta_1 t = (a\alpha u\beta v)\alpha_1 s\beta_1 t = a\alpha(u\beta v)\alpha_1(s\beta_1 t)$ ∈ a Γ T ΓT ⊆ a ∪ a Γ T ΓT. Therefore $r\alpha_1 s\beta_1 t$ ∈ a∪aΓTΓT and hence a∪aΓTΓT is a right ideal of T. Let R be a right ideal of T containing a.

Let r ∈ a∪aΓTΓT. Then $r = a$ or $r = a\alpha_2 u\beta_2 v$ for some u, v∈T; α_2, β_2 ∈ Γ. If $r = a$ then $r = a$ ∈ R. If $r = a\alpha_2 u\beta_2 v$ then $r = a\alpha_2 u\beta_2 v$ ∈ R. Therefore a∪aΓTΓT ⊆ R and hence a∪aΓTΓT is the smallest right ideal containing a. Therefore L(a)= a∪aΓTΓT.

We now introduce a right simple ternary Γ-semigroup and characterize right simple ternary Γ-semigroups.

Definition 3. 12:
A ternary Γ-semigroup T is said to be a right simple ternary Γ-semigroup if T is its only right ideal

Theorem 3. 13:
A ternary Γ-semigroup T is a right simple ternary Γ-semigroup if and only if $a\Gamma T\Gamma T = T$ for all $a \in T$.

Proof:
Suppose that T is a right simple ternary Γ-semigroup and $a \in T$. Let $s \in a\Gamma T\Gamma T$; t, $u \in T$. $s \in a\Gamma T\Gamma T \Rightarrow s = a\alpha v\beta w$ where v, $w \in T$ and $\alpha, \beta \in \Gamma$. Now $s\alpha_1 u\beta_1 t = (a\alpha v\beta w)\alpha_1 u\beta_1 t = a\alpha(v\beta w)\alpha_1(u\beta_1 t) \in a\Gamma T\Gamma T \Rightarrow a\Gamma T\Gamma T$ is a right ideal of T. Since T is a right simple ternary Γ-semi group; $a\Gamma T\Gamma T = T$. Therefore $a\Gamma T\Gamma T = T$ for all $a \in T$. Conversely suppose that $a\Gamma T\Gamma T = T$ for all $a \in T$. Let R be a right ideal of T. Let $r \in R$ then $r \in T$. By assumption $r\Gamma T\,\Gamma T = T$. Let $t \in T$ then $t \in r\Gamma T\Gamma T$ implies $t = r\alpha_1 u\beta_1 v$ for some $u, v \in T$ and $\alpha_1, \beta_1 \in \Gamma, r \in R$; $u, v \in T$; $\alpha_1, \beta_1 \in \Gamma$ and R is a right ideal$\Rightarrow r\alpha_1 u\beta_1 v \in R$; implies $t \in R$. Therefore $T \subseteq R$. Clearly $R \subseteq T$ and hence $R = T$. i. e T is the only right ideal of T. Hence T is a right simple ternary Γ-semigroup.

References:

1) Anjaneyulu. A; Structure and ideal theory of semigroups, Thesis, ANU(1980)
2) Borowiee. A; W. Adudek; S. Duplij; "Basic concepts of Ternary hop algebras, Math. QA, (2003), 1-11.
3) Clifford. A. H and Preston. G. B; the algebraic theory of semi groups. Vol-I, Americal Mathematical Society, providence (1961).
4) Kasner. E; An extension of the group concept, Bull. Amer. Math. Soc. 10(1904), 290-291
5) Lehmer. D. H; A ternary analogue of abelian groups, Amer. J. Math., 59(1932)329-338.
6) Lalitha. G; Sarala. Y; A note on right ideals of ordered semigroups. GJPAM(2014), 707-710
7) Nareupanat Lekkoksung and Prakit Jampachon: On regular ordered ternary semigroups; Far East Journal of Mathematical sciences. Vol-92(2014), 67-84.
8) Sarala. Y; Anjaneyulu. A and Madhusudhana Rao. D; ideals in ternary semigroups international ejournal of mathematics and engineering 203(2013), 1950-1968.
9) Sen. M. k, Sahan. K-On Γ-Semigroup. 1., Bull. Calcutta math. Soc-78(1986), 180-186.