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On the p-supersolubility of one class of finite groups

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Abstract

Let G be a finite group and H a subgroup of G. Then H is said to be *τ-quasinormal in G* if H permutes with all Sylow subgroups Q of G such that $(|Q|, |H|) = 1$ and $(|H|, |Q^G|) \neq 1$. Our main result here is the following: *Let $G = AT$, where A is a Hall π-subgroup of G and T is p-nilpotent for some prime $p \notin \pi$, let P denote a Sylow p-subgroup of T and assume that A is τ-quasinormal in G. Suppose that there is a number p^k such that $1 < p^k < |P|$ and A permutes with every subgroup of P of order p^k and with every cyclic subgroup of P of order 4 (if $p^k = 2$ and P is non-abelian). Then G is p-supersoluble.*

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1. Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. The symbol $\pi(n)$ denotes the set of all primes dividing $|n|$; $\pi(G) = \pi(|G|)$.

If H is a subgroup of G, then H^G denotes the normal closure of H in G, that is, the intersection of all normal subgroups of G containing H. Recall that a subgroup A of a group G is said to *permute* with a subgroup B if $AB = BA$. A subgroup H of G is said to be $\pi(G)$*-permutable* or $\pi(G)$*-quasinormal* in G (O. Kegel, [13]) if H permutes with all Sylow subgroups of G. A subgroup H of G is said to be *τ-quasinormal in G* (V.O. Lukyanenko and A.N. Skiba [16]) if H permutes with all Sylow subgroups Q of G such that $(|Q|, |H|) = 1$ and $(|H|, |Q^G|) \neq 1$. It is clear that every $\pi(G)$-quasinormal subgroup is τ-quasinormal. The Example 1.2 in [16] shows that the converse does not hold in general.

By a well known Hall's theorem [9], G is soluble if every Sylow subgroup P of G has a complement T in G, that is a subgroup of G such that $PT = G$ and $P \cap T = 1$. The example of the alternating group A_5 shows that such a result is incorrect in general if we consider only the Sylow p-subgroups for some fixed p. Nevertheless, B. Huppert [11] proved that if a Sylow p-subgroup P of G has a complement T in G, $|P| > p$ and T permutes with every maximal subgroup of P, then G is p-soluble. This result was improved in some directions. V.I. Sergienko [17] on base of this result proved that if a Sylow p-subgroup P of G has a complement T in G and there is a number p^k such that $1 < p^k < |P|$ and T permutes with all subgroups of P of order p^k and P is abelian in the case $p^k = 2$, then G is p-soluble and the p-length of G is equal to 1. Further, M.T. Borovikov [2] proved that under these conditions, G is even p-supersoluble. In [8] W. Guo, K.P. Shum and A.N. Skiba proved that if $G = AT$, where A is a Hall π-subgroup of G, T is nilpotent, and A permutes with all Sylow subgroups of T and with all maximal subgroups of any Sylow subgroup of T, then G is p-supersoluble, for each prime $p \notin \pi$ such that $|P| > p$ for a Sylow p-subgroup P of T.

In this paper, we prove the following generalization of all these results.

Theorem A. Let $G = AT$, where A is a Hall π-subgroup of G and T is a p-nilpotent for some prime $p \notin \pi$ subgroup of G. Let P be a Sylow p-subgroup of T. If A is τ-quasinormal in G and there is a number k such that $1 < p^k < |P|$ and A permutes with every subgroup of P of order p^k and with every cyclic subgroup of P of order 4 (if $p^k = 2$ and P is non-abelian), then G is p-supersoluble.

From Theorem A we obtain the following corollaries.

Corollary 1.1. (V.I. Sergienko [17, Theorems 2 and 3]). Let $G = PA$, where P is a Sylow p-subgroup of G and A is a Hall p'-subgroup of G. Suppose that there is a number p^k such that $1 < p^k < |P|$ and every subgroup of P of order p^k permutes with A and if $p^k = 2$, P is abelian. Then G is p-soluble and $l_p(G) = 1$.

Corollary 1.2. (M.T. Borovikov [2, Theorem]). Let $G = PA$, where P is a Sylow p-subgroup of G and A is a Hall p'-subgroup of G. Suppose that there is a number p^k such that $1 < p^k < |P|$ and every subgroup of P of order p^k permutes with A and if $p^k = 2$, P is abelian. Then G is p-supersoluble.

Corollary 1.3. (W. Guo, A.N. Skiba [6, Theorem C]). Let $G = AT$, where A is a Hall π-subgroup of G and T is a nilpotent subgroup of G. If A permutes with all Sylow subgroups of T and with all maximal subgroups of any Sylow subgroup of T, then G is p-supersoluble, for each prime $p \notin \pi$ such that T contains a Sylow p-subgroup P with $|P| > p$.

Corollary 1.4. (W. Guo, K.P. Shum, and A.N. Skiba [8, Theorem 4.6]). Let $G = AT$, where A is a Hall π-subgroup of G and T is a minimal nilpotent supplement of A in G. If A permutes with all maximal subgroups of any Hall subgroup of T, then G is p-supersoluble, for each prime $p \notin \pi$ such that T contains a Sylow p-subgroup P with $|P| > p$.

Corollary 1.5. (V.O. Lukyanenko, A.N. Skiba [14, Theorem 1.1]). Let $G = AT$, where A is a Hall π-subgroup of G and T is p-nilpotent for some prime $p \notin \pi$, let P denote a Sylow p-subgroup of T and assume that A permutes with every Sylow subgroup of T. Suppose that there is a number p^k such that $1 < p^k < |P|$ and A permutes with every subgroup of P of order p^k and with every cyclic subgroup of P of order 4 (if $p^k = 2$ and P is non-abelian). Then G is p-supersoluble.

As application of Theorem A we also prove the following fact.

Theorem B. Let $G = AT$, where A is a Hall π-subgroup of G and T is p-supersoluble for some prime $p \notin \pi$, and assume that A is τ-quasinormal in G. If A permutes with every maximal subgroup of a Sylow p-subgroup P of T, then G is p-supersoluble.

As a corollary of Theorem B we obtain the following result.

Corollary 1.6. (V.O. Lukyanenko, A.N. Skiba [14, Theorem 1.2]). Let $G = AT$, where A is a Hall π-subgroup of G and T is p-supersoluble for some prime $p \notin \pi$. Suppose that A permutes with every Sylow subgroup of T and with every maximal subgroup of a Sylow p-subgroup P of T. Then G is p-supersoluble.

The reader is referred to [1], [3] and [5] for all unexplained terminology and definitions if necessary.

2. Preliminaries

Lemma 2.1. (See [16, Lemma 2.2]). Let H be a normal subgroup of a group G. Then EH/H is τ-quasinormal in G/H for every τ-quasinormal subgroup E in G satisfying $(|H|, |E|) = 1$.

Lemma 2.2. (See [14, Lemma 2.1]). Let G be a group, N an elementary abelian normal p-subgroup of G for some prime p and A a Hall p'-subgroup of G. Suppose that there is a number p^k such that $1 < p^k < |N|$ and A permutes with every subgroup of N of order p^k. Then some maximal subgroup of N is normal in G.

Lemma 2.3. (See [14, Lemma 2.4]). Let $G = AP$ be a p-soluble group, where P is a Sylow p-subgroup of G and A is a Hall p'-subgroup of G. Suppose that there is a number p^k such that $1 < p^k < p^{k+1} < |P|$ and A permutes with every subgroup of P of order p^k and with every cyclic subgroup of P of order 4 (if $p^k = 2$ and P is non-abelian). Then $l_p(G) = 1$.

Recall that a formation is a homomorph $\mathcal{F}$ of groups such that each group G has the smallest normal subgroup (denoted by $G^{\mathcal{F}}$) whose quotient is still in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if it contains each group G with $G/\Phi(G) \in \mathcal{F}$.

Lemma 2.4. (See [14, Lemma 2.2]). Let $\mathcal{F}$ be a saturated formation containing all nilpotent groups and G a group with soluble $\mathcal{F}$-residual $P = G^{\mathcal{F}}$. Suppose that every maximal subgroup of G not containing P belongs to $\mathcal{F}$. Then P is a p-group for some prime p. In addition, if A is a Hall p'-subgroup of G and A permutes with all cyclic subgroups of P of prime order and order 4 (if $p = 2$ and P is non-abelian), then $|P/\Phi(P)| = p$.

Lemma 2.5. (See [14, Lemma 2.3]). Let V be a subgroup with order 4 of a group G and Q a Hall $2'$-subgroup of G such that $VQ = QV$.

(1) If $V = A \times B$, where $|A| = |B| = 2$ and $AQ = QA$, then $BQ = QB$.

(2) If $V = \langle x \rangle$ is cyclic, then $\langle x^2 \rangle Q = Q\langle x^2 \rangle$.

3. Proof of Theorems A and B

Proof of Theorem A. Assume that the theorem is false and let G be a counterexample of minimal order. We proceed the proof by proving the following claims.

(1) $O_{\pi_1}(G) = 1$, where $\pi_1 = \pi' \setminus \{p\}$.

Suppose that $Y = O_{\pi_1}(G) \neq 1$. We show that the hypothesis still holds on G/Y. In fact, $Y \cap P = 1$ and $G/Y = (AY/Y)(T/Y)$, where $AY/Y \simeq A$ is a Hall π-subgroup of G/Y, T/Y is p-nilpotent and $PY/Y \simeq P$ is a Sylow p-subgroup of T/Y. In view of Lemma 2.1, AY/Y is τ-quasinormal in G/Y. Let H^*/Y be a subgroup of PY/Y with $|H^*/Y| = p^k$. Then $H^* = [Y](H^* \cap P)$, where $H^* \cap P$ is a Sylow p-subgroup of H^* with $|H^* \cap P| = p^k$. By hypothesis, $A(H^* \cap P) = (H^* \cap P)A$. Hence AY/Y permutes with H^*/Y in G/Y. If $p^k = 2$ and PY/Y is non-abelian, then P is non-abelian and by hypothesis, A permutes with every cyclic subgroup of P of order 4. Hence similarly AY/Y permutes with every cyclic subgroup of PY/Y of order 4. Thus the hypothesis still holds on G/Y. Since $|G/Y| < |G|$, G/Y is p-supersoluble by the choice of G. It follows that G is p-supersoluble, a contradiction. Thus we have (1).

(2) A permutes with every Sylow q-subgroup of G, where $q \in \pi'$.

Let Q be a Sylow q-subgroup of G for some prime $q \in \pi'$. If $(|A|, |Q^G|) \neq 1$, then $AQ = QA$ by hypothesis. Suppose that $(|A|, |Q^G|) = 1$. Since $G = AT$, $Q^G \leq T$. Then Q^G is p-nilpotent. Since $O_{p'}(Q^G)Q^G G$, $O_{p'}(Q^G) \leq O_{\pi_1}(G) = 1$ in view of Claim (1). Therefore $Q^G = Q$, so $AQ = QA$. Thus A permutes with every Sylow q-subgroup of G, where $q \in \pi'$.

(3) $O_{p'}(G) = 1$.

Suppose that $V = O_{p'}(G) \neq 1$. We show that the hypothesis still holds on G/V. In fact, $V \cap P = 1$ and $G/V = (AV/V)(TV/V)$, where $AV/V \simeq A/V \cap A$ is a Hall π-subgroup of G/V, $TV/V \simeq T/V \cap T$ is p-nilpotent and $PV/V \simeq P$ is a Sylow p-subgroup of TV/V. Let Q/V be a Sylow q-subgroup of G/V, where $q \in \pi'$. Then for some Sylow q-subgroup G_q of G we have $Q = G_q V$. In view of Claim (2),

$$(AV/V)(Q/V) = (AV/V)(G_q V/V) = (Q/V)(AV/V),$$

so AV/V permutes with every Sylow q-subgroup of G/V, where $q \in \pi'$. Thus AV/V is τ-quasinormal in G/V. Besides, as above in the proof of (1) we have that AV/V permutes with every subgroup of PV/V of order p^k and with every cyclic subgroup of PV/V of order 4 (if $p^k = 2$ and PV/V is non-abelian). Thus the hypothesis still holds on G/V. Since $|G/V| < |G|$, G/V is p-supersoluble by the choice of G. It follows that G is p-supersoluble, a contradiction. Hence we have (3).

(4) $T = P$.

Suppose that $T \neq P$. Let S be a Hall p'-subgroup of T. Since A is a Hall π-subgroup of G, every Sylow subgroup of A is a Sylow subgroup of G. Then [18, Lemma 11.6] implies that A permutes with some Sylow q-subgroup of T, where $q \in \pi$. Hence in view of Claim (2), A permutes with every Sylow subgroup of T, so $AS = SA$. By hypothesis, S is normal in T. Then

$$S^G = S^{AT} = S^A \leq AS.$$

This means that $O_{p'}(G) \neq 1$, which contradicts Claim (3). Thus (4) holds.

(5) G is not a simple group.

Assume that G is a simple non-abelian group. By hypothesis, A permutes with every subgroup H of P with $|H| = p^k$. Since in view of Claim (4), $G = AP$, for any $x \in G$ we have $x = ta$, where $t \in P$ and $a \in A$. Then $H^t \leq P$. Hence $AH^t = H^t A$, so $(AH^t)^a = AH^x$ is a subgroup of G. Since $p \notin \pi$, $H \cap A = 1$. Therefore G is not simple by [12, Theorem 3]. This contradiction completes the proof of (5).

(6) $|P| > p^{k+1}$.

Assume that $|P| = p^{k+1}$. Then in view of [11, Theorem 5], G is p-soluble. Let L be a minimal normal subgroup of G. Then L is a p-group by (3), so $L \leq P$. We claim that G/L is p-supersoluble. Indeed, if $|P/L| \leq p$, it is evident. On the other hand, if $|P/L| > p$, then, clearly, the hypothesis still holds on G/L. Thus G/L is p-supersoluble by the choice of G. Obviously, $|L| > p$ and $L \nsubseteq \Phi(P)$. Let V be a maximal subgroup of P such that $L \nleq V$. Then $AV = VA$, so

$$D = V^G = V^{PA} = V^A \leq AV.$$

It follows from the facts that $P = VL$, V is maximal in P and $|L| \neq p$, that $V \cap L \neq 1$, therefore $D \cap L$ is a nontrivial subgroup of L and $D \cap L$ is normal in G, which contradicts the minimality of L. Hence (6) holds.

(7) $|N| \leq p^k$ for any minimal normal subgroup N of G contained in P.

Assume that $p^k < |N|$. Then A permutes with every subgroup H of N with $|H| = p^k$, so by Lemma 2.2 some maximal subgroup of N is normal in G, a contradiction. Thus we have (7).

(8) If N is a proper normal subgroup of G, then N is p-soluble.

In view of Claim (3), p divides $|N|$. First suppose that $A \leq N$. Then by Claim (4), $|G : N| = p^a$ for some $a \in \mathbb{N}$. Hence $N = N_p A$, where $N_p = N \cap P$ is a Sylow p-subgroup of N. Let P_1 be a maximal subgroup of P such that $N_p \leq P_1$. Then $P_1 N = P_1 A$ is a proper subgroup of G. In view of (6) the hypothesis still holds on $P_1 A$. Since $|P_1 A| < |G|$, $P_1 A$ is p-supersoluble by the choice of G. It follows that N is p-soluble. Now suppose that $A \not\leq N$. It is clear that $N = (A \cap N)(P \cap N)$. Let $E = (A \cap N)P$ and let H be a subgroup of P such that $AH = HA$. Then

$$AH \cap (A \cap N)P = (A \cap (A \cap N)P)H = (A \cap N)(A \cap P)H = (A \cap N)H = H(A \cap N).$$

This shows that the hypothesis still holds on E. If $E = G$, then

$$A = A \cap (A \cap N)P = (A \cap N)(A \cap P) = A \cap N,$$

a contradiction. Hence $E < G$, so E is p-supersoluble by the choice of G. It follows that $N \leq E$ is p-soluble.

(9) $k > 1$.

Suppose that $k = 1$. By Claim (5), G is not a simple group. Let $L = (A \cap L)(P \cap L)$ be a maximal normal subgroup of G. In view of Claim (8), L is p-soluble and by Claim (3), $L_p = P \cap L \neq 1$. Besides, since by Claim (3), $O_{p'}(G) = 1$ and $O_{p'}(L)LG$, $O_{p'}(L) = 1$. If L_p is cyclic or $|L_p| = p^2$, then $l_p(L) = 1$ by [10, VI, Hilfssatz 6.10]. Otherwise, since $k = 1$, the hypothesis still holds on L, so again $l_p(L) = 1$ by Lemma 2.3. Since $O_{p'}(L) = 1$, L_p is normal in L. Moreover, since L_p is a Sylow subgroup of L and L is characteristic in G, L_p is normal in G.

Suppose that $p = 2$. Then by Claim (6), $2^3 \leq |P|$. Since L is not 2-nilpotent, it has a 2-closed Schmidt subgroup [10, IV, Satz 5.4] of the form $K = [K_2]K_q$, where $K_2 \leq L_2$. For some $x \in G$, $K_q^x \leq A$ and $K^x = [K_2]^x K_q^x$. Since L_p is normal in G, $K_2^x \leq L_2 \leq P$. Let V be a cyclic subgroup of K_2^x of order 2 or order 4 (if K_2^x is non-abelian). Then $VA = AV$ by hypothesis, so

$$VA \cap K_2^x K_q^x = (VA \cap K_2^x)K_q^x = (V(A \cap K_2^x))K_q^x = VK_q^x = K_q^x V.$$

Thus K_q^x permutes with every cyclic subgroup of K_2^x of order 2 and order 4 (if K_2^x is non-abelian). Lemma 2.4 implies $|K_2^x/\Phi(K_2^x)| = 2$. It follows that $|K_2| = 2$. Hence K is nilpotent, contrary to the choice of K.

Thus $p > 2$. Let N be a minimal normal subgroup of G contained in L_p and $C = C_G(N)$. Then $|N| = p$ by Claim (7). Hence $N \leq Z(P)$. Assume that $C = G$. Let $a \in A$, $z \in N$ and $X = \langle x \rangle$ be any subgroup of L_p with $|X| = p$. Then $z^a = z$

and since $X = L_p \cap AX < AX$, $x^a = x^n$ for some $n \in \mathbb{N}$. Since $z \in Z(P)$, $|xz| = p$. Hence $\langle xz \rangle = \{x^i z^i \mid i = 1, \dots, p\}$ and

$$(xz)^a = x^a z^a = x^n z \in \langle xz \rangle,$$

whence $n = 1$. Therefore $A/C_A(L_p)$ acts trivially on $\Omega_1(L_p)$. Hence $A/C_A(L_p) = 1$ by [4, 5, Theorem 3.10], so

$$A \cap L \leq A = C_A(L_p) \leq C_G(L_p).$$

Hence $L = L_p(A \cap L)$ is a normal p-nilpotent subgroup of G, a contradiction. Therefore $C \neq G$. By Claim (8), C is p-soluble. Since $|N| = p$, G/C is cyclic and we deduce that G is p-soluble. Then in view of Lemma 2.3 and by Claims (3) and (6), P is normal in G. Let $R = G^{\mathcal{F}}$, where $\mathcal{F}$ is the saturated formation of all p-supersoluble groups and $\Phi = \Phi(R)$. It is clear that $R \leq P$. Then the hypothesis holds for every maximal subgroup M of G not containing R. Hence M is p-supersoluble by the choice of G, so $|R/\Phi| = p$ by Lemma 2.4. Then [19, Lemma 2.16] implies $G/\Phi \in \mathcal{F}$. But then $R \leq \Phi$, whence $R = \Phi$, a contradiction. Thus $k > 1$.

(10) If P is a non-abelian 2-group, then $k > 2$.

Suppose that $k = 2$. Since P is non-abelian, it has a cyclic subgroup $H = \langle x \rangle$ with $|H| = 4$. By hypothesis, A permutes with H. Then A permutes with $\langle x^2 \rangle$ by Lemma 2.5(2). Now note that if G has a subgroup $V = B \times C$ of order 4 such that $|B| = 2$ and A permutes with B, then A permutes with V and C. Indeed, since $|V| = 4$, A permutes with V. Hence A permutes with C by Lemma 2.5(1). Therefore A permutes with some subgroup Z of $Z(P)$ with $|Z| = 2$, so A permutes with every subgroup of P with order 2 by Lemma 2.5(1), which contradicts (9).

(11) If N is a minimal normal subgroup of G contained in P, then the hypothesis still holds on G/N.

In view of Claim (6), $|P| > p^{k+1}$. Besides, if either $p > 2$ and $|N| < p^k$ or $p = 2$ and $|N| < 2^{k-1}$, the hypothesis still holds on G/N. So let either $p > 2$ and $|N| = p^k$ or $p = 2$ and $|N| \in \{2^k, 2^{k-1}\}$. By Claim (9), $k > 1$. Suppose that $|N| = p^k$. Then N is non-cyclic and hence every subgroup of G containing N is non-cyclic. Let $N \leq K \leq P$, where $|K : N| = p$. Since K is non-cyclic, it has a maximal subgroup $F \neq N$. Then A permutes with $K = FN$, as K is the product of two subgroups permutable with A. Thus if $p > 2$ or P/N is abelian, the conclusion holds on G/N. Next suppose that P/N is a non-abelian 2-group. Then P is non-abelian, so $k > 2$ by Claim (10). Let $N \leq K \leq V$, where $|V : N| = 4$ and $|V : K| = 2$. Let K_1 be a maximal subgroup of V such that $V = K_1 K$. Suppose that K_1 is cyclic. Then $N \nleq K_1$, so $V = K_1 N$, which implies $|N| = 4$ or $|N| = 2$. But then $k = 2$ or $k = 1$, a contradiction. Hence K_1 is non-cyclic and as above we see that A permutes with K_1 and therefore with V. Thus again the hypothesis still holds on G/N. Now, suppose that $|N| = p^{k-1}$. If $|N| > 2$, then as above one can show that AN/N permutes with every cyclic subgroup of P/N

of order 2 and order 4 (if P/N is non-abelian). Finally, suppose that $|N| = 2$ and P/N is non-abelian. Then P is non-abelian and $k = 2$, which contradicts Claim (10). Hence we have (11).

Final contradiction.

By Claims (3), (6) and Lemma 2.3, $P = O_p(G)$. Let N be a minimal normal subgroup of G contained in P. Then by Claim (7), $N < P$. Since the class of all p-supersoluble groups is a saturated formation, in view of Claim (11), $N \not\subseteq \Phi(G)$ and N is the only minimal normal subgroup of G. Let M be a maximal subgroup of G such that $G = [N]M$. Then $P = P \cap NM = N(P \cap M)$. Since $P = F(G) \leq C_G(N)$, it follows that $P \cap M$ is normal in G, so $P \cap M = 1$. Hence $N = O_p(G) = P$. This contradiction completes the proof of the theorem. ■

Now we use Theorem A to prove Theorem B.

Proof of Theorem B. Assume that the theorem is false and let G be a counterexample of minimal order. We proceed the proof by proving the following claims.

(1) $O_{\pi_1}(G) = 1$, where $\pi_1 = \pi' \setminus \{p\}$ (see (1) in the proof of Theorem 1.1).

(2) $O_p(G) = 1$.

Suppose that $O_p(G) \neq 1$. Let L be a minimal normal subgroup of G contained in $O_p(G)$. Then $L \leq P$. We claim that G/L is p-supersoluble. Indeed, if $|P/L| \leq p$, it is evident. On the other hand, if $|P/L| > p$, then the hypothesis still holds on G/L in view of Lemma 2.1. Thus G/L is p-supersoluble by the choice of G. Obviously, $|L| > p$ and $L \not\subseteq \Phi(T)$. Let V be a maximal subgroup of T such that $T = LV$. Since T is p-supersoluble, $|T : V| = p$ by [10, VI, Satz 9.2(a)]. Then $1 \neq L \cap V < T$. Let $V_p = V \cap P$. Then V_p is maximal in P. By hypothesis, $AV_p = V_pA$, so

$$L \cap V = L \cap V_p = L \cap AV_p < AV_p.$$

Hence $A \leq N_G(L \cap V)$. It follows that $L \cap V$ is a nontrivial subgroup of L and $L \cap V$ is normal in G, which contradicts the minimality of L. Hence we have (2).

(3) A permutes with every Sylow q-subgroup of G, where $q \in \pi'$.

Let Q be a Sylow q-subgroup of G for some prime $q \in \pi'$. If $(|A|, |Q^G|) \neq 1$, then $AQ = QA$ by hypothesis. Suppose that $(|A|, |Q^G|) = 1$. Since $G = AT$, $Q^G \leq T$. Then Q^G is p-supersoluble. Hence either $O_p(Q^G) \neq 1$ or

$$O_{p'}(Q^G) = O_{\pi_1}(Q^G) \neq 1.$$

Since $O_p(Q^G)$ and $O_{p'}(Q^G)$ are characteristic subgroups of Q^G, we have a contradiction by Claims (1) or (2). Thus we have (3).

(4) $O_{p'}(G) = 1$ (see Claim (3) in the proof of Theorem A).

Final contradiction.

Since $p \in \pi'$, $AP = PA$ by Claim (3). Then in view of Theorem A, $G \neq AP$ and AP is p-supersoluble. Now suppose that $O_{p'}(T) \neq 1$. Let Q be a Sylow q-subgroup of T such that $q \neq p$ and $Y = O_q(T) \neq 1$. If $q \in \pi$, then [18, Lemma 11.6] implies that A permutes with Q. Otherwise, $AQ = QA$ by Claim (3). Hence

$$Y^G = Y^{AT} = Y^A \leq AQ.$$

This means that $O_{p'}(G) \neq 1$, which contradicts to Claim (4). Thus $O_{p'}(T) = 1$ and by [7, Lemma 3.3], T is supersoluble. Hence p is the largest prime divisor of $|T|$, so P is normal in T. Thus

$$P^G = P^{AT} = P^A \leq AP,$$

so P^G is p-supersoluble. Since by (4), $O_{p'}(G) = 1$ and $O_{p'}(P^G) < P^G < G$, $O_{p'}(P^G) = 1$. Then by [7, Lemma 3.3], P^G is supersoluble. Therefore P is normal in G, which contradicts to Claim (2). This completes the proof of the theorem. ■

Proof of Corollary 1.6. Let G_q be any Sylow q-subgroup of G for some prime $q \in \pi'$. Then for some Sylow q-subgroup Q of T we have $G_q = Q^x$, where $x \in G$. Since $G = TA$, we have $x = ta$, where $t \in T$ and $a \in A$. Hence $Q^x = Q^{ta} = (Q^t)^a$, so

$$AG_q = AQ^x = A(Q^t)^a = (AQ^t)^a = (Q^tA)^a = Q^xA = G_qA.$$

Thus A permutes with every Sylow q-subgroup of G, where $q \in \pi'$. Hence A is τ-quasinormal in G, so G is p-supersoluble by Theorem B. This completes the proof. ■

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