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Poisson Approximation for Random Sums of Independent Yule Random Variables

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Abstract

This paper determines a bound for the total variation distance between the distribution of random sums of independent Yule random variables and an appropriate Poisson distribution.

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1. Introduction

Let $X_1, \ldots, X_n$ be independently distributed Yule random variables, each with probability $P(X_i = k) = \dfrac{a_i k!\Gamma(a_i + 1)}{\Gamma(a_i + k + 2)}, k \in \mathbb{N}\cup\{0\}$. Suppose that N is a positive integer-valued random variable and independent of X_i's. Let $\mathbf{S}_N = \sum_{i=1}^{N} X_i$ be the random sums of independent Yule random variables and $\mathbf{P}_\lambda$ denote the the Poisson random variable with mean λ. For $N = n, n \in \mathbb{N}$, is fixed, Teerapabolarn [2] gave a bound for the total variation distance between the distributions of $\mathbf{S}_n$ and $\mathbf{P}_{\lambda_n}$ with mean $\lambda_n = \sum_{i=1}^{n} \dfrac{1}{a_i - 1}$ as follows:

$$d(\mathbf{S}_n, \mathbf{P}_{\lambda_n}) \leq \sum_{i=1}^{n} \frac{3a_i - 1}{(a_i^2 - 1)(a_i - 1)}, \tag{1.1}$$

where $d(\mathbf{S}_n, \mathbf{P}_{\lambda_n}) = \sup_{A \subseteq \mathbb{N}\cup\{0\}} |P(\mathbf{S}_n \in A) - P(\mathbf{P}_{\lambda_n} \in A)|$.

Let $\lambda_N = \sum_{i=1}^{N} \frac{1}{a_i - 1}$ and $\lambda = E(\lambda_N)$. In this study, we are interested to determine a bound for $d(\mathbf{S}_N, \mathbf{P}_\lambda)$, which is in Section 2. In Section 3, we give two examples to illustrate the desired result, and conclusion of this study is also presented in the last section.

2. Result

Before giving the main result, we also need the following lemma, which is directly obtained from [1].

Lemma 2.1. With the above definitions, we have the following:

$$d(\mathbf{P}_{\lambda_N}, \mathbf{P}_\lambda) \le \min\left\{1, \sqrt{\frac{2}{\lambda e}}\right\} E|\lambda_N - \lambda|. \tag{2.1}$$

The following theorem presents a bound for $d_{TV}(\mathbf{S}_N, \mathbf{P}_\lambda)$, which is the desired result.

Theorem 2.2. The following inequality holds:

$$\begin{aligned} d(\mathbf{S}_N, \mathbf{P}_\lambda) \le E\left(\frac{1 - e^{-\lambda_N}}{\lambda_N} \sum_{i=1}^{N} \frac{3a_i - 1}{(a_i^2 - 1)(a_i - 1)}\right) \\ + \min\left\{1, \sqrt{\frac{2}{\lambda e}}\right\} E|\lambda_N - \lambda|. \end{aligned} \tag{2.2}$$

Proof. From (1.1), (2.1) and using Lemma 2.1, it follows the fact that

$$\begin{aligned} d(\mathbf{S}_N, \mathbf{P}_\lambda) &\le \sum_{n=1}^{\infty} P(N = n) d(\mathbf{S}_n, \mathbf{P}_\lambda) \\ &\le \sum_{n=1}^{\infty} P(N = n) d(\mathbf{S}_n, \mathbf{P}_{\lambda_n}) + d(\mathbf{P}_{\lambda_N}, \mathbf{P}_\lambda) \\ &\le \sum_{n=1}^{\infty} P(N = n) \frac{1 - e^{-\lambda_n}}{\lambda_n} \sum_{i=1}^{n} \frac{3a_i - 1}{(a_i^2 - 1)(a_i - 1)} + d(\mathcal{P}_{\lambda_N}, \mathcal{P}_\lambda) \\ &\le E\left(\frac{1 - e^{-\lambda_N}}{\lambda_N} \sum_{i=1}^{N} \frac{3a_i - 1}{(a_i^2 - 1)(a_i - 1)}\right) + \min\left\{1, \sqrt{\frac{2}{\lambda e}}\right\} E|\lambda_N - \lambda|. \end{aligned}$$

Hence, the inequality (2.2) holds. ∎

If X_i's are identically distributed, then the following corollary is an immediately consequence of the Theorem 2.2.

Corollary 2.3. If $a_i = a$ for every i, then $\lambda_n = \dfrac{n}{a-1}$ and we have the following:

$$d(\mathbf{S}_N, \mathbf{P}_\lambda) \le E(1 - e^{-\frac{N}{a-1}})\frac{3a-1}{a^2-1} + \min\left\{1, \sqrt{\frac{2(a-1)}{E(N)e}}\right\} \frac{E|N - E(N)|}{a-1}.$$

3. Examples

This section gives two examples to illustrate the result in the case of X_i's to be identically distributed.

Example 3.1. For n $(n \in \mathbb{N})$ is fixed, let N be a random variable with probability function

$$P(N = k) = \begin{cases} \frac{1}{2} & , k = n, \\ \frac{1}{2} & , k = 2n, \\ 0 & , \text{otherwise.} \end{cases}$$

Therefore, $E(N) = \dfrac{3n}{2}$ and $E|N - E(N)| = \dfrac{n}{2}$. Let $a_i = a$ for every i, then $\lambda = \dfrac{3n}{2(a-1)}$ and we have

$$d(\mathbf{S}_N, \mathbf{P}_\lambda) \le \frac{3a-1}{a^2-1} + \min\left\{1, \sqrt{\frac{4(a-1)}{3ne}}\right\} \frac{n}{2(a-1)}.$$

Example 3.2. Let N be a random variable with probability function

$$P(N = n) = \frac{1}{2^n}, \; n = 1, 2, \ldots,$$

then we have $E(N) = 2$ and $E|N - E(N)| = 1$. If $a_i = a$ for every i, then $\lambda = \dfrac{2}{a-1}$ and we can obtain

$$d(\mathbf{S}_N, \mathbf{P}_\lambda) \le \frac{3a-1}{a^2-1} + \frac{1}{a-1} \min\left\{1, \sqrt{\frac{a-1}{e}}\right\}.$$

4. Conclusion

In this study, a bound for the total variation distance between the distribution of the random sums of independent Yule random variables and a Poisson distribution was obtained. With this bound, it can be seen that the result gives a good Poisson approximation when all a_i are large.

References

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