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Pointwise Poisson Approximation for Independent Negative Binomial Random Variables

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Abstract

This paper gives a non-uniform bound for approximating the probability function of a sum of independent negative binomial random variables and the Poisson probability function with mean $\sum_{i=1}^{n} \frac{r_i q_i}{p_i}$, where r_i and $p_i = 1 - q_i$ are parameters of each negative binomial distribution. It indicates that the probability function of the sum can be approximated by the Poisson probability function when all $r_i q_i$ are small.

AMS subject classification: 60F05, 60G05.
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1. Introduction

Let $X_1, \ldots, X_n$ be n independently distributed negative binomial random variables, each with probability $P(X_i = k) = \frac{\Gamma(r_i + k)}{\Gamma(r_i)k!} q_i^k p_i^{r_i}$, $k \in \mathbb{N} \cup \{0\}$, mean $\mu_i = \frac{r_i q_i}{p_i}$ and variance $\sigma_i^2 = \frac{r_i q_i}{p_i^2}$ where $q_i = 1 - p_i$. Let $\mathbf{S}_n = \sum_{i=1}^{n} X_i$ and $\mathbf{P}_\lambda$ denote the the Poisson random variable with mean λ. Note that, if all $r_i q_i$ are small, then the distribution of $\mathbf{S}_n$ is approximately a Poisson distribution with mean λ. For $\lambda = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} r_i q_i$ and

$A \subseteq \mathbb{N} \cup \{0\}$, Vellaisamy and Upadhye [6] gave a bound in the form of

$$d_A(\mathbf{S}_n, \mathbf{P}_\lambda) \leq \sum_{i=1}^{n} \frac{r_i q_i^2}{p_i} \min\left\{1, \frac{1}{\sqrt{2\lambda e}}\right\}, \tag{1.1}$$

where $d_A(\mathbf{S}_n, \mathbf{P}_\lambda) = |P(\mathbf{S}_n \in A) - P(\mathbf{P}_\lambda \in A)|$ is the distance between the distribution of $\mathbf{S}_n$ and the Poisson distribution. For $\lambda = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \frac{r_i q_i}{p_i}$, Teerapabolarn [5] gave a bound as follows:

$$d_A(\mathbf{S}_n, \mathbf{P}_\lambda) \leq \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{n} \frac{r_i q_i^2}{p_i^2}. \tag{1.2}$$

However, for $A = \{x_0; x_0 \in \mathbb{N} \cup \{0\}\}$, the result in (1.2) becomes

$$d_{x_0}(\mathbf{S}_n, \mathbf{P}_\lambda) = |P(\mathbf{S}_n = x_0) - P(\mathbf{P}_\lambda = x_0)| \leq \frac{1 - e^{-\lambda}}{\lambda} \sum_{i=1}^{n} \frac{r_i q_i^2}{p_i^2} \tag{1.3}$$

for every x_0. It is observed that the bound is a uniform constant for the point metric $d_{x_0}(\mathbf{S}_n, \mathbf{P}_\lambda)$. In this situation, a non-uniform bound with respect to x_0 is required. In this paper, we are interested to derived a non-uniform bound for $d_{x_0}(\mathbf{S}_n, \mathbf{P}_\lambda)$ by using the same tools in [4], which are described in Section 2. In Section 3, we use the Stein-Chen method and the negative binomial w-functions to derive the desired result and the conclusion in this study is presented in the last section.

2. Method

The following lemma is also need to prove the main result.

Lemma 2.1. For $1 \leq i \leq n$, let w_i be the w-function associated with the negative binomial random variable X_i, then we have the following:

$$w_i(k) = \frac{(r_i + k) q_i}{p_i \sigma_i^2}, \; k \in \mathbb{N} \cup \{0\} \;\; [3]. \tag{2.1}$$

The following relation is an important property for proving the result, which was stated by [2].

$$\begin{aligned} Cov(\mathbf{S}_n, f(\mathbf{S}_n)) &= \sum_{i=1}^{n} Cov\left(X_i, f\left(X_i + \sum_{j \neq i} X_j\right)\right) \\ &= \sum_{i=1}^{n} \sigma_i^2 E[w_i(X_i) \Delta f(\mathbf{S}_n)], \end{aligned} \tag{2.2}$$

for any function $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ for which $E|w_i(X_i)\Delta f(\mathbf{S}_n)| < \infty$, where $\Delta f(x) = f(x+1) - f(x)$.

For the Stein-Chen method, following [1], it is applied for every constant $\lambda > 0$, for every $x_0 \in \mathbb{N} \cup \{0\}$ and the bounded real valued function $f = f_{\{x_0\}} : \mathbb{N} \cup \{0\} \to \mathbb{R}$. Thus, Stein's equation for the Poisson distribution with these conditions is of the form

$$P(\mathbf{S}_n = x_0) - P(\mathbf{P}_{\lambda_n} = x_0) = E[\lambda f(\mathbf{S}_n + 1) - \mathbf{S}_n f(\mathbf{S}_n)]. \tag{2.3}$$

For $x_0 \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{N}$, [3] showed that

$$\sup_{k \geq 1} |f(k+1) - f(k)| \leq \min\left\{\frac{1 - e^{-\lambda}}{\lambda}, \frac{1}{x_0}\right\}. \tag{2.4}$$

3. Result

The following theorem gives a non-uniform bound for the metric $d_{x_0}(\mathbf{S}_n, \mathbf{P}_\lambda)$.

Theorem 3.1. Let $\lambda = \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$. Then the following inequality holds:

$$d_{x_0}(\mathbf{S}_n, \mathbf{P}_\lambda) \leq \min\left\{\frac{1 - e^{-\lambda}}{\lambda}, \frac{1}{x_0}\right\} \sum_{i=1}^{n} \frac{r_i q_i^2}{p_i^2}. \tag{3.1}$$

Proof. From (2.3), it follows that

$$\begin{aligned}
d_{x_0}(\mathbf{S}_n, \mathbf{P}_\lambda) &= |\lambda E[f(\mathbf{S}_n + 1)] - E[\mathbf{S}_n f(\mathbf{S}_n)]| \\
&= |\lambda E[\Delta f(\mathbf{S}_n)] - Cov(\mathbf{S}_n, f(\mathbf{S}_n))| \\
&= \left| \sum_{i=1}^{n} \mu_i E[\Delta f(\mathbf{S}_n)] - \sum_{i=1}^{n} Cov(X_i, f(\mathbf{S}_n))] \right|.
\end{aligned}$$

Using (2.2) and Lemma 1, we have

$$\begin{aligned}
d_{x_0}(\mathbf{S}_n, \mathbf{P}_\lambda) &= \left| \sum_{i=1}^{n} \left\{ E\left[\mu_i \Delta f(\mathbf{S}_n)\right] - \sigma_i^2 E[w_i(X_i)\Delta f(\mathbf{S}_n)] \right\} \right| \\
&\leq \sum_{i=1}^{n} E\{|r_i q_i/p_i - \sigma_i^2 w_i(X_i)||\Delta f(\mathbf{B}_n)|\} \\
&\leq \sup_{k \geq 1} |\Delta f(k)| \sum_{i=1}^{n} E\,|r_i q_i/p_i - (r_i + X_i) q_i/p_i| \\
&\leq \sup_{k \geq 1} |\Delta f(k)| \sum_{i=1}^{n} \frac{r_i q_i^2}{p_i^2}.
\end{aligned}$$

Hence, by (2.4), (3.1) is obtained. ■

Corollary 3.2. For $r_1 = r_2 = \cdots = r_n = 1$, then $\lambda = \sum_{i=1}^{n} \frac{q_i}{p_1}$ and

$$d_{x_0}(\mathbf{S}_n, \mathbf{P}_\lambda) \leq \min\left\{\frac{1-e^{-\lambda}}{\lambda}, \frac{1}{x_0}\right\} \sum_{i=1}^{n} \frac{q_i^2}{p_i^2}. \tag{3.2}$$

The result (3.2) is a Poisson approximation for a sum of independent geometric random variables. When all X_i are identically distributed random variables, thus immediately from the Theorem 1, we have the following Corollary.

Corollary 3.3. If $r_1 = r_2 = \cdots = r_n = r$ and $p_1 = p_2 = \cdots = p_n = p$, then $\lambda = \frac{nrq}{p}$ and the following inequality holds:

$$d_{x_0}(\mathbf{S}_n, \mathbf{P}_\lambda) \leq \min\left\{1-e^{-\lambda}, \frac{\lambda}{x_0}\right\} \frac{q}{p}. \tag{3.3}$$

4. Conclusion

In this study, a non-uniform bound on the point metric between the probability function of a sum of independent negative binomial random variables and the Poisson probability function with mean $\sum_{i=1}^{n} \frac{r_i q_i}{p_i}$ was obtained. With this bound, it is seen that the probability function of the summands can be approximated by the Poisson probability function when $r_i q_i$ is small for every $i \in \{1, \ldots, n\}$.

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