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Pointwise Negative Binomial Approximation to The Beta Binomial Distribution

K. Teerapabolarn
Department of Mathematics,
Faculty of Science,
Burapha University,
Chonburi 20131, Thailand.
E-mail: kanint@buu.ac.th

Abstract

This paper determines a non-uniform bound on pointwise approximation of the beta binomial distribution with parameters n, α and β by a negative binomial distribution with parameters α and $\frac{\alpha+\beta}{\alpha+\beta+n}$. It indicates that the result gives a good approximation when β is large.

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Keywords: Beta binomial distribution, negative binomial approximation, Stein's method, beta binomial w-function.

1. Introduction

Let X be the beta binomial random variable with parameters $n \in \mathbb{N}$, $\alpha > 0$, and $\beta > 0$. It has the probability function as follows:ss

$$p_X(x) = \binom{n}{x} \frac{B(x+\alpha, n-x+\beta)}{B(\alpha, \beta)}, \ x = 0, 1, \ldots, n, \tag{1.1}$$

and has mean $\mu = \frac{n\alpha}{\alpha+\beta}$ and variance

$$\sigma^2 = \frac{n\alpha\beta(n+\alpha+\beta)}{(\alpha+\beta)^2(1+\alpha+\beta)},$$

where B is the complete beta function. It is observed that $\alpha+\beta\to\infty$ while $\dfrac{n}{\alpha+\beta}$ remains a constant, then the beta binomial distribution with parameters n, α and β converges to a negative binomial distribution with parameters α and $p=\dfrac{\alpha+\beta}{\alpha+\beta+n}$. Therefore, it is reasonable to approximate the beta binomial distribution by the negative binomial with these parameters when $\alpha+\beta$ is large. In this case, Teerapabolarn [3] gave a bound for the total variation distance between the negative binomial and beta binomial distributions as follows:

$$d_A(X,Y)\le(1-p^{\alpha})\frac{(\alpha+1)(\alpha+\beta+n)}{(\alpha+\beta)(\alpha+\beta+1)}, \tag{1.2}$$

where $d_A(X,Y)=|P(X\in A)-P(Y\in A)|$ for $A\subseteq\mathbb{N}\cup\{0\}$ and Y is the negative binomial random variable with parameters α and p. However, for $A=\{x_0;\,x_0\in\mathbb{N}\cup\{0\}\}$ and $d_{x_0}(X,Y)=|P(X=x_0)-P(Y=x_0)|$, the result in (1.2) becomes

$$d_{x_0}(X,Y)\le(1-p^{\alpha})\frac{(\alpha+1)(\alpha+\beta+n)}{(\alpha+\beta)(\alpha+\beta+1)} \tag{1.3}$$

for every x_0. It is observed that the bound is a uniform constant for the point metric $d_{x_0}(X,Y)$. In view of this situation, a non-uniform bound with respect to x_0 is required. In this paper, we focus on deriving a non-uniform bound for $d_{x_0}(X,Y)$ in (1.3) by using Stein's method and the beta binomial w-function, which are in Section 2. In Section 3, we derive the desired result of this study, and the conclusion of this study is presented in the last section.

2. Method

The following lemma is the w-function associated with the beta binomial random variable X, which is directly obtained from [3].

Lemma 2.1. We have

$$w(x)=\frac{(n-x)(\alpha+x)}{(\alpha+\beta)\sigma^2},\ x=0,1,\dots,n, \tag{2.1}$$

where

$$\sigma^2=\frac{n\alpha\beta(n+\alpha+\beta)}{(\alpha+\beta)^2(1+\alpha+\beta)}.$$

The next relation is an important property to obtain the desired result, which was stated by [2]. If a non-negative integer-valued random variable X has $p_X(x)>0$ for every x in support of X and $0<\sigma^2=\mathbb{V}ar(X)<\infty$, then

$$\mathbb{C}ov(X,g(X))=\sigma^2\mathbb{E}[w(X)\Delta g(X)], \tag{2.2}$$

for any function $g : \mathbb{N} \cup \{0\} \to \mathbb{R}$ for which $\mathbb{E}|w(X)\Delta g(X)| < \infty$, where $\Delta g(x) = g(x+1) - g(x)$ and $\mathbb{E}[w(X)] = 1$.

For Stein's method in the negative binomial approximation, it can be applied for $\alpha > 0$ and $0 < p = 1-q < 1$, for every $x_0 \in \mathbb{N} \cup \{0\}$ and bounded real-valued function $g = g_{x_0} : \mathbb{N} \cup \{0\} \to \mathbb{R}$ defined as in Brown and Phillips [1]. So, Stein's equation for these conditions is as follows:

$$P(X = x_0) - P(Y = x_0) = E[q(\alpha + X)g(X+1) - Xg(X)]. \tag{2.3}$$

For $x_0 \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{N}$, [4] showed that

$$\sup_{x \geq 1} |\Delta g(x)| \leq \delta(x_0) = \begin{cases} \dfrac{1 - p^{\alpha}}{\alpha q} & \text{if } x_0 = 0, 1, \\ \min\left\{\dfrac{1}{x_0}, \dfrac{1 - p^{\alpha}}{(\alpha + x_0 - 1)q}\right\} & \text{if } x_0 > 1. \end{cases} \tag{2.4}$$

3. Result

The following theorem gives a non-uniform bound in pointwise negative binomial approximation to the beta binomial distribution.

Theorem 3.1. For $\alpha > 0$, $p = 1 - q = \dfrac{\alpha + \beta}{\alpha + \beta + n}$ and $x_0 \in \mathbb{N} \cup \{0\}$, then

$$d_{x_0}(X, Y) \leq \frac{\delta(x_0)\alpha(\alpha + 1)n}{(\alpha + \beta)(\alpha + \beta + 1)}. \tag{3.1}$$

Proof. From (2.3), we have

$$\begin{aligned} P(X = x_0) - P(Y = x_0) &= \mathbb{E}[\alpha q g(X+1) + qX\Delta g(X) - pXg(X)] \\ &= \alpha q \mathbb{E}[g(X+1)] + q\mathbb{E}[X\Delta g(X)] - p\mathbb{E}[Xg(X)] \\ &= \alpha q \mathbb{E}[\Delta g(X)] + q\mathbb{E}[X\Delta g(X)] - p\mathbb{C}ov(X, g(X)). \end{aligned}$$

Using Lemma 2.1 and (2.2), we have $E|w(X)\Delta g(X)| < \infty$. From which it follows that

$$\begin{aligned} d_{x_0}(X, Y) &= \left|\alpha q \mathbb{E}[\Delta g(X)] + q\mathbb{E}[X\Delta g(X)] - p\mathbb{E}[\sigma^2 w(X)\Delta g(X)]\right| \\ &\leq \mathbb{E}\{|(\alpha + X)q - \sigma^2 w(X)p||\Delta g(X)|\} \\ &= \mathbb{E}\left\{\frac{(\alpha + X)X}{\alpha + \beta + n}|\Delta g(X)|\right\} \ (\geq 0). \end{aligned}$$

We obtain

$$\begin{aligned} d_{x_0}(X, Y) &\leq \mathbb{E}\{[(r + X)q - \sigma^2 w(X)p]|\Delta g(X)|\} \\ &= (\mu - \sigma^2 p)\mathbb{E}|\Delta g(X)| \\ &\leq \delta(x_0)(\mu - \sigma^2 p) \\ &\leq \frac{\delta(x_0)\alpha(\alpha + 1)n}{(\alpha + \beta)(\alpha + \beta + 1)}, \end{aligned}$$

which completes the proof. ■

4. Conclusion

In the present study, a non-uniform bound for the point metric between the beta binomial and negative binomial distributions is derived by using Stein's method and the w-function associated with the beta binomial random variable. With this bound, it is observed that the result in Theorem 3.1 gives a good approximation when β is large, that is, the negative binomial distribution with parameters α and $\frac{\alpha+\beta}{\alpha+\beta+n}$ can be used as an approximation of the beta binomial distribution with parameters n, α and β when β is large with respect to n and α.

References

[1] Brown, T. C., Phillips, M. J., 1999, "Negative binomial approximation with Stein's method", Meth. Comp. Appl. Probab., 1(4), 407–421.

[2] Cacoullos, T., Papathanasiou, V., 1989, "Characterization of distributions by variance bounds", Statist. Probab. Lett., 7(5), 351–356.

[3] Teerapabolarn, K., 2015, "Negative binomial approximation to the beta binomial distribution", Int. J. Pure Appl. Math., 98(1), 39–43.

[4] Malingam, P., Teerapabolarn, K., 2011, "A pointwise negative binomial approximation by w-functions", Int. J. Pure Appl. Math., 71(1), 57–66.