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Finite Queueing Model with arrival rate depends upon some factors and having more spares than servers

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Abstract

This paper is devoted to the analysis of finite queueing model with arrival rate depends upon spares, reneging and balking. We discuss the model under Markovian environment. Steady state probability vector is obtained using NETUS matrix method.

AMS subject classification:
Keywords: Queueing models, finite queue, infinitesimal generator, balking, reneging and spares.

1. Introduction

Medhi [2] analysed many stochastic models in queueing theory. Gross and Haris [1] discussed fundamentals for queueing theory. Mitchell and Paulson [3] discussed M/M/1 queues with independent arrival and service process. Neuts [6, 7] analyzed M/M/1 queue with randomly varying arrival and service rates and explicit steady state solutions to some elementary queueing models. Usha [10] discussed PH/M/C queue with varying environment. Ramanarayanan and Usha [9] discussed Markovian queueing systems with two waiting rooms. Neuts [8] discussed matrix-geometric solutions in stochastic model.

Murthy and Ramanarayanan [4, 5] discussed two ordering levels inventory system with different lead times and rest time to the server using NETUS matrix method and two (s, S) inventory systems with binary choice of demands and optional accessories using NETUS matrix method.

In this paper we find the solution of the finite queueing model with arrival rate depends upon some factors viz balking, reneging or spares. The steady state probability p_n, that there are n units in the system the probability of an empty system p_o and the expected number of units in the system are derived.

We assume that we have a finite source of N customers, each with an arriving rate λ, C servers are available, that the service times are identical exponential random variables with rate μ and also the system has finite storage room such that the total number of customers in the system is at most K. We assume that we have Y spares on hand so that when machine fails, a spare is immediately substituted for it. If it happens that all spares are used and a breakdown occurs, then the system becomes short. When a machine is repaired, then it becomes a spare.

Let $\beta =$ prob. (a unit joins the queue)

$$\beta = 1, \text{ if } n \leq \min(Y, C)$$

and

$$0 \leq \beta < 1 \text{ if } \begin{cases} Y + 1 \leq n \leq Y + K \text{ (When Y is minimum)} \\ C + 1 \leq n \leq Y + K \text{ (When C is minimum)} \end{cases}$$

Let α be σ rate of reneging, having an exponential distribution given by

$$f(t) = \alpha e^{-\alpha t}; \; t, \; \alpha \geq 0. \tag{1.1}$$

A unit, after being in the queue for service a certain time t (a random variable), may leave it without being served, with probability given by

$$g(n) = \text{probability (a unit reneges)}$$

clearly

$$g(n) = (n - C)\alpha; \; C + 1 \leq n \leq \gamma + K. \tag{1.2}$$

Now we discuss the case $Y \geq C$ and one can find the results for the case $Y < C$. Here we assume that spares $Y \geq C$. Then the set of birth death coefficients are as follows:

$$\lambda_n = \begin{cases} (N - n)\lambda & ; 0 \leq n \leq C \\ (N - n)\beta\lambda & ; C + 1 \leq n \leq Y \\ (N - n)\beta\lambda w & ; Y + 1 \leq n \leq Y + K - 1 \\ 0 & ; n \geq Y + K \end{cases} \tag{1.3}$$

and

$$\mu_n = \begin{cases} n\mu & ; 0 \leq n \leq C \\ C\mu - (n - C)\alpha & ; C + 1 \leq n \leq Y + K. \end{cases} \tag{1.4}$$

The infinitesimal generator Q of continuous time Markovian chain is as follows:

	0	1	2	3	4	…	C-1	C	C+1	C+2	…	Y-1	Y	Y+1	Y+2	…	Y+K-1	Y+K
0	$-N\lambda$	$N\lambda$																
1	μ	$-(N-1)\lambda -\mu$	$(N-1)\lambda$															
2		2μ	$-(N-2)\lambda -2\mu$	$(N-2)\lambda$														
3			3μ	$-(N-3)\lambda -3\mu$	$(N-3)\lambda$	…												
⋮						⋮												
⋮						⋮												
C-1							$-(N-C+1)\lambda -(C-1)\mu$	$(N-C+1)\lambda$										
C							$C\mu$	$-(N-C)\lambda -C\mu$	$(N-C)\lambda$									
C+1								$C\mu+\alpha$	$-(N-C-1)\beta\lambda -(C\mu+\alpha)$	$(N-C-1)\beta\lambda$	…							
C+2									$C\mu+2\alpha$	$-(N-C-2)\beta\lambda -(C\mu+2\alpha)$	…							
⋮									⋮	⋮								
⋮									⋮	⋮								
Y-1											…	$-(N-Y+1)\beta\lambda -(C\mu+(Y-1-C)\alpha)$	$(N-Y+1)\beta\lambda$					
Y												$C\mu+(Y-C)\alpha$	$-(N-Y)\beta\lambda -(C\mu+(Y-C)\alpha$	$(N-Y)\beta\lambda$				
Y+1													$C\mu+(Y+1-C)\alpha$	$-(N-Y-1)\beta\omega\lambda -(C\mu+(Y+1-C)\alpha)$	$(N-Y-1)\beta\omega\lambda$	…		
Y+2														$C\mu+(Y+2-C)\alpha$	$-(N-Y-2)\beta\omega\lambda -(C\mu+(Y+2-C)\alpha)$	…		
⋮														⋮	⋮			
⋮														⋮	⋮			
Y+K-1																…	$-(N-Y-K+1)\beta\omega\lambda -(C\mu+(Y+K-1-C)\alpha$	$(N-Y-K+1)\beta\omega\lambda$
Y+K																	$C\mu+(Y+K-C)\alpha$	$-(C\mu+(Y+K-C)\alpha)$

Let $\underline{\Pi}$ be the steady state probabilities of Q. Then $\underline{\Pi}Q = 0$ and $\underline{\Pi}\ \underline{e} = 1$ where $\underline{e} = (1, 1, 1, \dots, 1)^t$.

The steady state probability difference equations are

$$\begin{aligned}
&-N\lambda P_0 + \mu P_1 = 0;\ \ n = 0\\
&(N-n+1)\lambda P_{n-1} - [(N-n)\lambda + n\mu]P_n + (n+1)\mu P_{n+1} = 0;\ \ 1 \le n \le C\\
&-\{(N-n)\lambda\beta + (n-1)\mu - (n-C)\alpha\}\, P_n + [N-n+1]\lambda P_{n-1}\\
&+[(n-C)\mu + (n-C+1)\alpha]P_{n+1} = 0;\ \ C+1 \le n \le Y\\
&-\{(N-n)\lambda\beta w + C\mu + (n-C)\alpha\}\, P_n + [N-n+1]\lambda\beta P_{n-1}\\
&+[C\mu + (n-1-C)\alpha]P_{n+1} = 0;\ \ Y+1 \le n \le Y+K\\
&-\{C\mu + (n-C)\alpha\}\, P_n + [N-n+1]\lambda\beta w P_{n-1} = 0;\ \ n = Y+K
\end{aligned} \tag{1.5}$$

First we find the probabilities for $0 \le n \le C$. By (1.5) we have

$$-N\lambda P_0 + \mu P_1 = 0$$

From this we get

$$P_1 = \binom{N}{1}\left(\frac{\lambda}{\mu}\right)^1 P_0$$

Next we find P_2. By (1.5)

$$N\lambda P_0 - \{(N-n)\lambda + \mu\}\, P_1 + 2\mu P_2 = 0$$

$2\mu P_2 = [(N-1)\lambda + \mu]\, P_1 - N\lambda P_0$

$P_2 = \binom{N}{2}\left(\frac{\lambda}{\mu}\right)^2 P_0$

Therefore

$$P_n = \binom{N}{n}\left(\frac{\lambda}{\mu}\right)^n P_0 \ \text{ where }\ 0 \le n \le C.$$

Next we find the probabilities for $C+1 \le n \le Y$. For this first we find P_{C+1}. By (1.5)

$$(N+C+1)\lambda P_{C-1} - \{(N-C)\lambda + Cmu\}\, P_C + [C\mu + \alpha]P_{C+1} = 0$$

This implies

$$[C\mu + \alpha]P_{C+1} = \{(N-C)\lambda + Cmu\}\, P_C - (N+C+1)\lambda P_{C-1}$$

From this we get

$$P_{C+1} = \binom{N}{C}\left(\frac{\lambda}{\mu}\right)^{C+1} \beta^0 (N-C)\frac{1}{C+\frac{\alpha}{\mu}} P_0$$

Next we find P_{C+2}. By (1.5)

$$(N-C)\lambda P_C - \{(N-C-1)\lambda\beta + Cmu + \alpha\}\, P_{C+1} + [C\mu + 2\alpha] P_{C+2} = 0$$

$$[C\mu + 2\alpha] P_{C+2} = \{(N-C-1)\lambda\beta + Cmu + \alpha\}\, P_{C+1} - (N-C)\lambda P_C$$

From this we get,

$$P_{C+2} = \left(\begin{array}{c} N \\ C \end{array}\right)\left(\frac{\lambda}{\mu}\right)^{C+2} \beta^1 (N-C)(N-C-1)\frac{1}{(C+\frac{\alpha}{\mu})(C+\frac{2\alpha}{\mu})} P_0$$

Therefore for $C+1 \le n \le Y$, we have

$$P_n = \left(\begin{array}{c} N \\ C \end{array}\right)\left(\frac{\lambda}{\mu}\right)^{n} \beta^{n-c-1}(N-C)(N-(C+1))\dots(N-(n-1))$$
$$\frac{1}{(C+\frac{\alpha}{\mu})(C+\frac{2\alpha}{\mu})\dots(C+\frac{(n-C)\alpha}{\mu})} P_0$$

Next we find the probabilities for $Y+1 \le n \le Y+K$. First we find P_{Y+1}. By (1.5), we have

$$(N-\lambda+1)\lambda\beta P_{Y-1} - \{(N-\lambda)\lambda\beta + C\mu + (Y-C)\alpha\}\, P_Y$$
$$+ [C\mu + (Y+1-C)\alpha] P_{Y+1} = 0$$

This implies

$$\begin{aligned}
&[C\mu + (Y+1-C)\alpha] P_{Y+1} \\
&\quad = \{(N-\lambda)\lambda\beta + C\mu + (Y-C)\alpha\}\, P_Y - (N-\lambda+1)\lambda\beta P_{Y-1} \\
&\quad = \{(N-\lambda)\lambda\beta + C\mu + (Y-C)\alpha\} \left(\begin{array}{c} N \\ C \end{array}\right)\left(\frac{\lambda}{\mu}\right)^{Y} \beta^{Y-C-1} \\
&\qquad \times \frac{(N-C)[N-(C+1)]\dots[N-(Y-1)]}{\left(C+\frac{\alpha}{\mu}\right)\dots\left(C+\frac{(Y-C)\alpha}{\mu}\right)} P_0 \\
&\qquad - (N-Y+1)\lambda\beta \left(\begin{array}{c} N \\ C \end{array}\right)\left(\frac{\lambda}{\mu}\right)^{Y-1} \beta^{Y-C-2}
\end{aligned}$$

$$\times \frac{(N-C)[N-(C+1)]\dots[N-(Y-2)]}{\left(C+\frac{\alpha}{\mu}\right)\dots\left(C+\frac{(Y-C-1)\alpha}{\mu}\right)} P_0$$

From this we get

$$P_{Y+1} = \left(\begin{array}{c} N \\ C \end{array}\right)\left(\frac{\lambda}{\mu}\right)^{Y+1} \beta^{Y-C-1} w^0$$
$$\frac{(N-C)[N-(C+1)]\dots[N-(Y-1)][N-Y]}{\left(C+\frac{\alpha}{\mu}\right)\dots\left(C+\frac{(Y-C+1)\alpha}{\mu}\right)} P_0$$

Finally we find P_{Y+K}. By (1.5)

$$(N-Y-K+1)\lambda\beta w P_{Y+K-1} - [C\mu + (Y+K-C)\alpha]P_{Y+K} = 0$$

This implies

$$[C\mu + (Y+K-C)\alpha]P_{Y+K} = (N-Y-K+1)\lambda\beta w P_{Y+K-1}$$

From this we get

$$\begin{aligned} P_{Y+K} &= \frac{1}{\mu\left[C + \frac{(Y+K-C)\alpha}{\mu}\right]}(N-Y-K+1)\lambda\beta \\ &\quad \binom{N}{C}\left(\frac{\lambda}{\mu}\right)^{Y+K-1} \beta^{Y+K-C-3} w^{k} \\ &\quad \frac{(N-C)\ldots[N-(Y+K-2)]}{\left(C+\frac{\alpha}{\mu}\right)\ldots\left(C+\frac{(Y+K-C-1)\alpha}{\mu}\right)} P_0 \\ &= \binom{N}{C}\left(\frac{\lambda}{\mu}\right)^{Y+K} \beta^{Y+K-C-2} w^{K+1} \\ &\quad \frac{(N-C)\ldots[N-(Y+K-1)]}{\left(C+\frac{\alpha}{\mu}\right)\ldots\left(C+\frac{(Y+K-C)\alpha}{\mu}\right)} P_0 \end{aligned}$$

Therefore

$$P_n = \binom{N}{C}\left(\frac{\lambda}{\mu}\right)^{n} \beta^{n-C-2} w^{n-y-1} \frac{(N-C)\ldots[N-(n-1)]}{\left(C+\frac{\alpha}{\mu}\right)\ldots\left(C+\frac{(n-C)\alpha}{\mu}\right)} P_0,$$

For $Y+1 \le n \le Y+K$. Therefore the probabilities P_n are given by

$$P_n = \begin{cases} \binom{N}{n}\left(\frac{\lambda}{\mu}\right)^{n} P_0 & ; 0 \le n \le C \\ \binom{N}{C}\left(\frac{\lambda}{\mu}\right)^{n} \beta^{n-C-1} \frac{(N-C)\ldots[N-(n-1)]}{\left(C+\frac{\alpha}{\mu}\right)\ldots\left(C+\frac{(n-C)\alpha}{\mu}\right)} P_0 & ; C+1 \le n \le Y \\ \binom{N}{C}\left(\frac{\lambda}{\mu}\right)^{n} \beta^{n-C-2} w^{n-y-1} \frac{(N-C)\ldots[N-(n-1)]}{\left(C+\frac{\alpha}{\mu}\right)\ldots\left(C+\frac{(n-C)\alpha}{\mu}\right)} P_0 & ; Y+1 \le n \le Y+K. \end{cases}$$

Let $[N-C]_{(n)} = (N-C)[N-(C+1)]\ldots[N(n-1)]$ with this notation, the probabilities are given by

$$P_n = \begin{cases} \binom{N}{n}\left(\frac{\lambda}{\mu}\right)^n P_0 & ; 0 \le n \le C \\ \binom{N}{C}\left(\frac{\lambda}{\mu}\right)^n \beta^{n-C-1}\frac{[N-C]_{(n)}}{\prod_{i=1}^{n-C}\left(C+\frac{i\alpha}{\mu}\right)} P_0 & ; C+1 \le n \le Y \\ \binom{N}{C}\left(\frac{\lambda}{\mu}\right)^n \beta^{n-C-2} w^{n-Y-1}\frac{[N-C]_{(n)}}{\prod_{i=1}^{n-C}\left(C+\frac{i\alpha}{\mu}\right)} P_0 & ; Y+1 \le n \le Y+K. \end{cases}$$

Now we find P_0. The boundary condition is

$$\sum_{n=0}^{Y+K} P_n = 1$$

This implies

$$\sum_{n=0}^{C} P_n + \sum_{n=C+1}^{Y} P_n + \sum_{n=Y+1}^{Y+K} P_n = 1$$

This implies

$$P_0\left(\frac{\lambda}{\mu}\right)^n \left(\sum_{n=0}^{C}\binom{N}{n} + \binom{N}{C}\sum_{n=C+1}^{Y}\beta^{n-C-1}\frac{[N-C]_{(n)}}{\prod_{i=1}^{n-C}\left(C+\frac{i\alpha}{\mu}\right)} + \binom{N}{C}\sum_{n=Y+1}^{Y+K}\beta^{n-C-2}w^{n-Y-1}\frac{[N-C]_{(n)}}{\prod_{i=1}^{n-C}\left(C+\frac{i\alpha}{\mu}\right)}\right) = 1$$

Therefore

$$P_0 = \left(\frac{\lambda}{\mu}\right)^{-n} \left(\sum_{n=0}^{C}\binom{N}{n} + \binom{N}{C}\sum_{n=C+1}^{Y}\beta^{n-C-1}\frac{[N-C]_{(n)}}{\prod_{i=1}^{n-C}\left(C+\frac{i\alpha}{\mu}\right)} + \binom{N}{C}\sum_{n=Y+1}^{Y+K}\beta^{n-C-2}w^{n-y-1}\frac{[N-C]_{(n)}}{\prod_{i=1}^{n-C}\left(C+\frac{i\alpha}{\mu}\right)}\right)^{-1}$$

Now we find the average number of customers in the system. Average queue length is given by

$$\begin{aligned}
E(m) &= \sum_{n=C+1}^{Y+K} (n-C)P_n \\
&= \sum_{n=C+1}^{Y} (n-C)P_n + \sum_{n=Y+1}^{Y+K} (n-C)P_n \\
&= \sum_{n=C+1}^{Y} (n-C)\left(\begin{array}{c} N \\ C \end{array}\right)\left(\frac{\lambda}{\mu}\right)^n \beta^{n-C-1} \frac{[N-C]_{(n)}}{\prod_{i=1}^{n-C}\left(C+\frac{i\alpha}{\mu}\right)} P_0 \\
&\quad + \sum_{n=Y+1}^{Y+K} (n-C)\left(\begin{array}{c} N \\ C \end{array}\right)\left(\frac{\lambda}{\mu}\right)^n \beta^{n-C-1} w^{n-Y-1} \frac{[N-C]_{(n)}}{\prod_{i=1}^{n-C}\left(C+\frac{i\alpha}{\mu}\right)} P_0
\end{aligned}$$

Next,

$$E(m) = \sum_{n=C+1}^{Y+K} (n-C)P_n = \sum_{n=C+1}^{Y+K} nP_n - \sum_{n=C+1}^{Y+K} CP_n$$

Average number of customers in the system is

$$\begin{aligned}
E(n) &= \sum_{n=0}^{Y+K} nP_n \\
&= \sum_{n=0}^{C} nP_n + \sum_{n=C+1}^{Y+K} nP_n \\
&= \sum_{n=0}^{C} nP_n + E(m) + C\sum_{n=C+1}^{Y+K} P_n \\
&= \sum_{n=0}^{C} nP_n + E(m) + C\left(\sum_{n=0}^{X+k} nP_n - \sum_{n=0}^{C} P_n\right)
\end{aligned}$$

$$
\begin{aligned}
&=\sum_{n=0}^{C} nP_n + E(m) + C\left(1 - \sum_{n=0}^{C} P_n\right)\\
&=E(m) + C + \sum_{n=0}^{C} nP_n - C\sum_{n=0}^{C} P_n\\
&=E(m) + C - \sum_{n=0}^{C}(C-n)P_n\\
&=E(m) + C - \sum_{n=0}^{C}(C-n)\binom{N}{n}\left(\frac{\lambda}{\mu}\right)^n P_0\\
&=E(m) + C - P_0\sum_{n=0}^{C}(C-n)\binom{N}{n}\left(\frac{\lambda}{\mu}\right)^n
\end{aligned}
$$

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