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On A Certain Class Of Multivalent Uniformly Convex Functions Using Differintegral Operator

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Abstract

In this paper, using differintegral operator, we introduce a new class of multivalent uniformly convex functions in the unit disc $U=\{z:|z|<1\}$ and obtain the coefficient bounds, extreme bounds and radius of starlikeness for the functions belonging to this generalized class. Furthermore, partial sums $f_k(z)$ of functions $f(z)$ in the class $k-\mathrm{TUCV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ are considered. The various results obtained in this paper are sharp.

Key words and phrases: multivalent function, starlikeness, convexity, radius theorems, uniformly convex, uniformly starlike functions.

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1. INTRODUCTION AND DEFINITION

Let A_p be the class of functions analytic in the open unit disc $U=\{z:|z|<1\}$ of the form

$$f(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}z^{n+p}\ \ (p\geq 1) \tag{1.1}$$

and let $A_1 = A$.

A function $f(z)\in A_p$ is said to be k - uniformly p - valent starlike of order δ $(-p<\delta<p)$, $k\geq 0$ and $z\in U$ denoted by k – UST(p, δ) if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-\delta\right\}\geq k\left|\frac{zf'(z)}{f(z)}-p\right|.$$

A function $f(z)\in A_p$ is said to be k - uniformly p - valent convex of order δ $(-p<\delta<p)$, $k\geq 0$ and $z\in U$ denoted by k – UCV(p, δ) if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}-\delta\right\}\geq k\left|1+\frac{zf''(z)}{f'(z)}-p\right|.$$

For the functions $f(z)$ of the form (1.1) and $g(z)=z^p+\sum_{n=1}^{\infty}b_{n+p}z^{n+p}$, the hadamard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^p+\sum_{n=1}^{\infty}a_{n+p}b_{n+p}\,z^{n+p}.$$

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that the function $f(z)$ is subordinate to $g(z)$ in U, if there exists an analytic function $w(z)$ in U such that $|w(z)|<|z|$ and $f(z)=g(w(z))$, denoted by $f(z)\prec g(z)$. If $g(z)$ is univalent in U, then the subordination is equivalent to $f(0)=g(0)$ and $f(U)\subset g(U)$.

Recently, M.K. Aouf et. al. [1] introduced the operator $\Re_{\beta,p}^{\alpha,\gamma}:A_p\to A_p$ as follows:

$$\begin{aligned}\Re_{\beta,p}^{\alpha,\gamma}f(z)&=\frac{\Gamma(p+\alpha+\beta-\gamma+1)}{\Gamma(p+\beta)}\frac{1}{z^p}\int_0^z\left(1-\frac{t}{z}\right)^{\alpha-\gamma}t^{\beta-1}f(t)\,dt\\&=z^p+\frac{\Gamma(p+\alpha+\beta-\gamma+1)}{\Gamma(p+\beta)}\sum_{n=1}^{\infty}\left[\frac{\Gamma(p+\beta+n)}{\Gamma(p+\alpha+\beta+n-\gamma+1)}\right]a_{n+p}z^{n+p}\end{aligned}\qquad(1.2)$$

$(\beta>-p;\alpha>\gamma-1;\gamma\in\square;p\in\square;z\in U)$.

From (1.2), it is easy to verify that

$$z\left(\Re_{\beta,p}^{\alpha+1,\gamma}f(z)\right)'=(\alpha+\beta+p-\gamma+1)\Re_{\beta,p}^{\alpha,\gamma}f(z)-(\alpha+\beta-\gamma+1)\Re_{\beta,p}^{\alpha+1,\gamma}f(z).$$

By specifying the parameters, we obtain the following subclasses which were studied by various authors:

1. For $\gamma=1$, (1.3)

$$\begin{aligned}\Re_{\beta,p}^{\alpha,\gamma}f(z)&=Q_{\beta,p}^{\alpha}f(z)\\&=z^p+\frac{\Gamma(p+\alpha+\beta)}{\Gamma(p+\beta)}\sum_{n=1}^{\infty}\left[\frac{\Gamma(p+\beta+n)}{\Gamma(p+\alpha+\beta+n)}\right]a_{n+p}z^{n+p}\end{aligned}$$

$(\beta>-p;\alpha>0;p\in\square;z\in U)$

where the operator $Q_{\beta,p}^{\alpha}$ was introduced and studied by Liu and Owa [8] and $Q_{\beta,1}^{\alpha}=Q_{\beta}^{\alpha}$ introduced and studied by Jung et.al.[5].

2. For $\alpha=\gamma$ and $\beta=c$ (1.4)

$$\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z}) = J_{c,p} f(\mathrm{z})$$
$$= z^p + \sum_{n=1}^{\infty}\left(\frac{c+p}{c+n+p}\right) a_{n+p} z^{n+p} \qquad (c>-p;\, p\in \mathbb{N};\, z\in U)$$

where the operator $J_{c,p} f(\mathrm{z})$ is the familiar integral operator, which was defined by Saitoh et. al. [11]. For other choices of α and β, the operator $\Re_{\beta,p}^{\alpha,\gamma}$ reduces to the familiar other well- known integral operators introduced and discussed by various authors [3, 7, 9].

Let $T_p(n)$ be the subclass of A_p, consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p\geq 1). \tag{1.5}$$

Motivated by the earlier investigations of Murugusundaramoorthy [10], Amsheri and Zharkova [2] and various authors [4, 6], we investigate, in the present paper, the various properties and characteristics of analytic multivalent functions belonging to the subclass $\mathrm{k-TUCV}_p(\lambda,\alpha,\beta,\gamma,\delta)$.

For $0\leq\lambda\leq 1, 0\leq\delta<1$ and $k\geq 0$, we let $\mathrm{k-UCV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ be the subclass of A_p consisting of functions of the form (1.1) and satisfying the inequality

$$\mathrm{Re}\left\{\frac{\left(1-\lambda+\frac{\lambda}{p}\right) z\left(\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})\right)' + \frac{\lambda}{p} z^2\left(\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})\right)''}{p(1-\lambda)\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z}) + \lambda z\left(\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})\right)'} - \delta\right\}$$
$$> k\left|\frac{\left(1-\lambda+\frac{\lambda}{p}\right) z\left(\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})\right)' + \frac{\lambda}{p} z^2\left(\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})\right)''}{p(1-\lambda)\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z}) + \lambda z\left(\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})\right)'} - 1\right|, \tag{1.6}$$

where $z\in U$, $\Re_{\beta,p}^{\alpha,\gamma}$ is given by (1.2).

We further $\mathrm{k-TUCV}_p(\lambda,\alpha,\beta,\gamma,\delta) = \mathrm{k-UCV}_p(\lambda,\alpha,\beta,\gamma,\delta)\cap T_p$.

Example 1: If $\lambda=0$, then (1.6) becomes

$$\mathrm{Re}\left\{\frac{1}{p}\frac{z\left(\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})\right)'}{\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})} - \delta\right\} > k\left|\frac{1}{p}\frac{z\left(\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})\right)'}{\Re_{\beta,p}^{\alpha,\gamma} f(\mathrm{z})} - 1\right|, \qquad (z\in U). \tag{1.7}$$

Example 2: If $\lambda=1$, then (1.6) becomes

$$\mathrm{UCT}(\alpha,\beta,\gamma,\delta,k)=\mathrm{Re}\left\{\frac{1}{p}\left[1+\frac{z\left(\Re_{\beta,p}^{\alpha,\gamma}f(z)\right)''}{\left(\Re_{\beta,p}^{\alpha,\gamma}f(z)\right)'}\right]-\delta\right\}>k\left|\frac{1}{p}\left[1+\frac{z\left(\Re_{\beta,p}^{\alpha,\gamma}f(z)\right)''}{\left(\Re_{\beta,p}^{\alpha,\gamma}f(z)\right)'}\right]-1\right|. \tag{1.8}$$

Example 3: If $\gamma=1$ and $f(z)$ is as defined in (1.3) is in $Q_{\beta,p}^{\alpha}(\lambda,\delta,k)$ then (1.6) becomes

$$\mathrm{Re}\left\{\frac{\left(1-\lambda+\frac{\lambda}{p}\right)z\left(Q_{\beta,p}^{\alpha}f(z)\right)'+\frac{\lambda}{p}z^2\left(Q_{\beta,p}^{\alpha}f(z)\right)''}{p(1-\lambda)Q_{\beta,p}^{\alpha}f(z)+\lambda z\left(Q_{\beta,p}^{\alpha}f(z)\right)'}-\delta\right\}$$
$$>k\left|\frac{\left(1-\lambda+\frac{\lambda}{p}\right)z\left(Q_{\beta,p}^{\alpha}f(z)\right)'+\frac{\lambda}{p}z^2\left(Q_{\beta,p}^{\alpha}f(z)\right)''}{p(1-\lambda)Q_{\beta,p}^{\alpha}f(z)+\lambda z\left(Q_{\beta,p}^{\alpha}f(z)\right)'}-1\right|. \tag{1.9}$$

Also, let $Q_{\beta,p}^{\alpha}(\lambda,\delta,k)\cap T_p=TQ_{\beta,p}^{\alpha}(\lambda,\delta,k)$.

Example 4: If $\alpha=\gamma,\ \beta=c$ and $f(z)$ is as defined in (1.4) is in $J_{c,p}(\lambda,\delta,k)$ then (1.6) becomes

$$\mathrm{Re}\left\{\frac{\left(1-\lambda+\frac{\lambda}{p}\right)z\left(J_{c,p}f(z)\right)'+\frac{\lambda}{p}z^2\left(J_{c,p}f(z)\right)''}{p(1-\lambda)J_{c,p}f(z)+\lambda z\left(J_{c,p}f(z)\right)'}-\delta\right\}$$
$$>k\left|\frac{\left(1-\lambda+\frac{\lambda}{p}\right)z\left(J_{c,p}f(z)\right)'+\frac{\lambda}{p}z^2\left(J_{c,p}f(z)\right)''}{p(1-\lambda)J_{c,p}f(z)+\lambda z\left(J_{c,p}f(z)\right)'}-1\right|. \tag{1.10}$$

The main object of this paper is to study the coefficient bounds, extreme points and radius of starlikeness for functions belong to the generalized class $\mathrm{k}-\mathrm{TUCV}_p(\lambda,\alpha,\beta,\gamma,\delta)$, employing the technique of Silverman [12] and also derive results for the Hadamard products of functions belonging to the $\mathrm{k}-\mathrm{TUCV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ using the techniques of Schild and Silverman [15].

2. COEFFICIENT ESTIMATES

In this section, we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $\text{k}-\text{UCV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ and $\text{k}-\text{TUCV}_p(\lambda,\alpha,\beta,\gamma,\delta)$.

Theorem 2.1: The function $f(z)$ defined by (1.1) is in the class $\text{k}-\text{UCV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ if

$$\sum_{n=1}^{\infty}(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)\left|a_{n+p}\right| \le p^2(1-\delta), \qquad (z\in U), \qquad (2.1)$$

where

$$\phi(n,\alpha,\beta,\gamma)=\frac{\Gamma(p+\alpha+\beta-\gamma+1)\Gamma(p+\beta+n)}{\Gamma(p+\beta)\Gamma(p+\alpha+\beta+n-\gamma+1)}, \qquad (2.2)$$

$0\le\lambda\le 1, -1\le\delta<1$ and $k\ge 0$.

Proof: Since $f(z)\in \text{k}-\text{UCV}_p(\lambda,\alpha,\beta,\gamma,\delta)$, it suffices to show that

$$k\left|\frac{\left(1-\lambda+\frac{\lambda}{p}\right)z\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)'+\frac{\lambda}{p}\mathrm{z}^2\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)''}{p(1-\lambda)\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})+\lambda z\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)'}-1\right|-$$

$$\mathrm{Re}\left\{\frac{\left(1-\lambda+\frac{\lambda}{p}\right)z\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)'+\frac{\lambda}{p}\mathrm{z}^2\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)''}{p(1-\lambda)\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})+\lambda z\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)'}-1\right\}\le 1-\delta.$$

We have,

$$k\left|\frac{\left(1-\lambda+\frac{\lambda}{p}\right)z\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)'+\frac{\lambda}{p}\mathrm{z}^2\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)''}{p(1-\lambda)\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})+\lambda z\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)'}-1\right|$$

$$-\ \mathrm{Re}\left\{\frac{\left(1-\lambda+\frac{\lambda}{p}\right)z\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)'+\frac{\lambda}{p}\mathrm{z}^2\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)''}{p(1-\lambda)\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})+\lambda z\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)'}-1\right\}$$

$$\le(1+\mathrm{k})\left|\frac{\left(1-\lambda+\frac{\lambda}{p}\right)z\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)'+\frac{\lambda}{p}\mathrm{z}^2\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)''}{p(1-\lambda)\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})+\lambda z\left(\Re_{\beta,p}^{\alpha,\gamma}f(\mathrm{z})\right)'}-1\right|$$

$$\leq (1+\mathrm{k})\frac{\sum_{n=1}^{\infty} n\left(\frac{p+n\lambda}{p}\right)\phi(n,\alpha,\beta,\gamma)\left|a_{n+p}\right|}{p-\sum_{n=1}^{\infty}(p+n\lambda)\phi(n,\alpha,\beta,\gamma)\left|a_{n+p}\right|}.$$

The last inequality above is bounded by $(1-\delta)$ if

$$\sum_{n=1}^{\infty}(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)\left|a_{n+p}\right| \leq p^2(1-\delta)$$

and hence the proof is complete.

Theorem 2.2: The function $f(z)$ defined by (1.5) is in the class $\mathrm{k}-TU\,\mathrm{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ if and only if

$$\sum_{n=1}^{\infty}(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)a_{n+p} \leq p^2(1-\delta), \qquad (z\in U), \qquad (2.3)$$

where $\phi(n,\alpha,\beta,\gamma)$ is given by (2.2).

Proof: In view of Theorem 2.1, we need only to prove the necessity.

Let $f(z)\in \mathrm{k}-TU\,\mathrm{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ and z is real, then by inequality (1.6)

$$\frac{1-\sum_{n=1}^{\infty}\left(\frac{p+n}{p}\right)\left(\frac{p+n\lambda}{p}\right)\phi(n,\alpha,\beta,\gamma)a_{n+p}\left|z^n\right|}{1-\sum_{n=1}^{\infty}\left(\frac{p+n\lambda}{p}\right)\phi(n,\alpha,\beta,\gamma)a_{n+p}\left|z^n\right|}-\delta$$

$$\geq \frac{k\sum_{n=1}^{\infty} n\left(\frac{p+n\lambda}{p}\right)\phi(n,\alpha,\beta,\gamma)a_{n+p}\left|z^n\right|}{1-\sum_{n=1}^{\infty}\left(\frac{p+n\lambda}{p}\right)\phi(n,\alpha,\beta,\gamma)a_{n+p}\left|z^n\right|}.$$

Letting $z\rightarrow 1$ along the real axis, we obtain the desired inequality (2.3). The result (2.3) is sharp for the function

$$f(z)=z^p-\frac{p^2(1-\delta)}{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}z^{n+p}. \qquad (p,n\in \mathbb{N}) \qquad (2.4)$$

Corollary 2.3: If $f\in \mathrm{k}-TU\,\mathrm{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$, then

$$a_{n+p} \leq \frac{p^2(1-\delta)}{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)} \qquad (p,n\in \mathbb{N}). \qquad (2.5)$$

Corollary 2.4: A necessary and sufficient condition for $f(z)$ of the form (1.8) to be in the class $U\,\mathrm{CT}_p(\alpha,\beta,\gamma,\delta,k)$ is that

$$\sum_{n=1}^{\infty}(p+n)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)\,a_{n+p}\le p^2(1-\delta),\qquad (z\in U)$$ where

$\phi(n,\alpha,\beta,\gamma)$ is given by (2.2).

Corollary 2.5: A necessary and sufficient condition for $f(z)$ of the form (1.9) to be in the class $\mathrm{TQ}_p(\lambda,\delta,k)$ is that

$$\sum_{n=1}^{\infty}(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,1)\,a_{n+p}\le p^2(1-\delta),\qquad (z\in U)$$ where

$\phi(n,\alpha,\beta,\gamma)$ is given by (2.2).

Corollary 2.6: A necessary and sufficient condition for $f(z)$ of the form (1.10) to be in the class $\mathrm{TJ}_p(\lambda,\delta,k)$ is that

$$\sum_{n=1}^{\infty}(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\left(\frac{c+p}{c+n+p}\right)a_{n+p}\le p^2(1-\delta),\qquad (z\in U)\ .$$ Put

$c=1$, this class is reduced into the class $TI^p(\lambda,\alpha,\beta)$. (see[10]).

3. CLOSURE THEOREMS

Let the function $f(z)\in T_p(n)$ defined by (1.5) and the function

$$f_i(z)=z^p-\sum_{n=1}^{\infty}a_{n+p,i}\,z^{n+p}\qquad \left(a_{n+p,i}\ge 0,\ p\ge 1,\ i=1,2\ and\ z\in U\right). \tag{3.1}$$

be in the class $T_p(n)$, then the class $T_p(n)$ is said to be convex if

$$h(z)=\rho f_1(z)+(1-\rho)\,f_2(z)\in T_p(n) \tag{3.2}$$

where $0\le\rho\le 1$.

Theorem 3.1: The class $\mathrm{k}-TU\,\mathrm{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ is convex set.

Proof: Let $f_i(z)$ defined by (3.1), be in the class $\mathrm{k}-TU\,\mathrm{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ then

$$\rho f_1(z)+(1-\rho)\,f_2(z)=z^p-\sum_{n=1}^{\infty}\left[\rho\,a_{n+p,1}+(1-\rho)a_{n+p,2}\right]z^{n+p}.$$

An easy computation with aid of Theorem 2.2 gives

$$\sum_{n=1}^{\infty}(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)\left[\rho\, a_{n+p,1}+(1-\rho)a_{n+p,2}\right]$$
$$\leq p^2\rho(1-\delta)+p^2(1-\rho)(1-\delta)$$
$$\leq p^2(1-\delta).$$

This completes the proof.

Theorem 3.2: Let

$$f_p(z)=z^p \text{ and } f_{n+p}(z)=z^p-\frac{p^2(1-\delta)}{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}z^{n+p}. \qquad (3.3)$$

Then $f(z)\in \text{k}-TU\,\text{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ if and only if it can be expressed in the form

$$f(z)=\sum_{n=0}^{\infty}c_{n+p}f_{n+p}(z), \qquad c_{n+p}\geq 0, \qquad \sum_{n=0}^{\infty}c_{n+p}=1. \qquad (3.4)$$

Proof: Suppose that

$$f(z)=\sum_{n=0}^{\infty}c_{n+p}f_{n+p}(z)=z^p-\frac{p^2(1-\delta)}{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}\;c_{n+p}\;z^{n+p}.$$

Then we get,

$$\sum_{n=1}^{\infty}\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)\,p^2(1-\delta)}{p^2(1-\delta)(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}\;c_{n+p}$$
$$=\sum_{n=1}^{\infty}c_{n+p}=1-c_p\leq 1.$$

By virtue of Theorem 2.2, this shows that $f(z)$ is in the class $\text{k}-TU\,\text{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$. Conversely, assume that $f(z)$ belongs to the class $\text{k}-TU\,\text{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$. Again by virtue of Theorem 2.2, we have

$$a_{n+p}\leq\frac{p^2(1-\delta)}{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}.$$

Next, setting

$$c_{n+p}\leq\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\;a_{n+p}.$$

and $c_p=1-\sum_{n=1}^{\infty}c_{n+p}$, we can readily see that $f(z)$ can be expressed precisely as in (3.4).

This completes the proof of Theorem3.2.

4. RADII OF CLOSE – TO – CONVEXITY, STARLIKENESS AND CONVEXITY

Now we provide the radii of p- valently close – to – convexity, starlikeness and convexity for the class $\text{k}-TU\,\text{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$.

Theorem 4.1: Let $f\in \text{k}-TU\,\text{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$. Then f is p – valently close – to – convex of order η $(0\le\eta<p)$ in $|z|<R_1$, where

$$R_1=\inf_{n\in\mathbb{N}}\left\{\left[\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\left(\frac{p-\eta}{p+n}\right)\right]^{\frac{1}{n}}\right\}. \tag{4.1}$$

The result is sharp with the extremal function given by (2.4).

Proof: Given $f\in \text{k}-TU\,\text{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ and f is close – to – convex of order η, we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|<p-\eta. \tag{4.2}$$

For the left hand side of (4.2), we have

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|=\left|\frac{p\ z^{p-1}-\sum_{n=1}^{\infty}(p+n)a_{n+p}z^{n+p-1}}{z^{p-1}}-p\right|\le\sum_{n=1}^{\infty}(p+n)a_{n+p}|z|^n.$$

The last expression is less than $(p-\eta)$ if

$$\sum_{n=1}^{\infty}(p+n)a_{n+p}|z|^n<p-\eta$$

which implies

$$\sum_{n=1}^{\infty}\frac{p+n}{p-\eta}a_{n+p}|z|^n<1.$$

Using the fact that $f\in \text{k}-TU\,\text{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$, if and only if

$$\sum_{n=1}^{\infty}\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}a_{n+p}\le 1.$$

We can say that (4.2) is true if

$$\frac{p+n}{p-\eta}a_{n+p}|z|^n\le\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}.$$

$$\Rightarrow|z|^n\le\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\left(\frac{p-\eta}{p+n}\right).$$

The last inequality leads us immediately to the disc $|z| < R_1$, where R_1 is given by (4.1).

Theorem 4.2: Let $f \in \text{k}-TU\,\text{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$. Then f is p – valently starlike of order η $(0 \le \eta < p)$ in $|z| < R_2$, where

$$R_2 = \inf_{n\in\mathbb{N}}\left\{\left[\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\left(\frac{p-\eta}{p+n-\eta}\right)\right]^{\frac{1}{n}}\right\}. \tag{4.3}$$

The result is sharp with the extremal function given by (2.4).

Proof: Given $f \in \text{k}-TU\,\text{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$ and f is starlike of order η, we have

$$\left|\frac{z\,f'(z)}{f(z)} - p\right| < p-\eta. \tag{4.4}$$

For the left hand side of (4.4), we have

$$\left|\frac{z\,f'(z)}{f(z)} - p\right| = \left|\frac{p\,z^p - \sum_{n=1}^{\infty}(p+n)a_{n+p}z^{n+p}}{z^p - \sum_{n=1}^{\infty}a_{n+p}z^{n+p}} - p\right| \le \frac{\sum_{n=1}^{\infty}n\,a_{n+p}|z|^n}{1-\sum_{n=1}^{\infty}a_{n+p}|z|^n}.$$

The last expression is less than $(p-\eta)$ if

$$\frac{\sum_{n=1}^{\infty}n\,a_{n+p}|z|^n}{1-\sum_{n=1}^{\infty}a_{n+p}|z|^n} < p-\eta$$

which implies

$$\sum_{n=1}^{\infty}\frac{p+n-\eta}{p-\eta}a_{n+p}|z|^n < 1.$$

Using the fact that $f \in \text{k}-TU\,\text{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$, if and only if

$$\sum_{n=1}^{\infty}\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}a_{n+p} \le 1.$$

We can say that (4.2) is true if

$$\frac{p+n-\eta}{p-\eta}|z|^n \le \frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}.$$

$$\Rightarrow |z|^n \le \frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\left(\frac{p-\eta}{p+n-\eta}\right).$$

The last inequality leads us immediately to the disc $|z| < R_2$, where R_2 is given by (4.3).

Theorem 4.3: Let $f \in \text{k}-TU\,\text{CV}_p(\lambda,\alpha,\beta,\gamma,\delta)$. Then f is p – valently convex of order η $(0 \le \eta < p)$ in $|z| < R_3$, where

$$R_3 = \inf_{n \in \mathbb{N}} \left\{ \left[\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)} \left(\frac{p(p-\eta)}{(p+n)(p+n-\eta)} \right) \right]^{\frac{1}{n}} \right\}. \tag{4.3}$$

The result is sharp with the extremal function given by (2.4).

The proof is omitted, since we use a similar proof of Theorem 4.2.

5. PARTIAL SUMS

In this we consider partial sums of functions in the class $\text{k}-TUCV_p(\lambda,\alpha,\beta,\gamma,\delta)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$, $f_m(z)$ and $f'(z)$ to $f'_m(z)$. Silverman [13] and Silvia [14] have studied the partial sums of analytic functions.

Theorem 5.1: Let $\text{k}-TUCV_p(\lambda,\alpha,\beta,\gamma,\delta)$ be given by (1.1) and define the partial sums of $f_1(z)$ to $f_m(z)$ by

$$f_1(z) = z^p \text{ and } f_m(z) = z^p + \sum_{n=1}^{m-1} a_{n+p} z^{n+p}, \quad m = 2,3,\cdots. \tag{5.1}$$

If

$$\sum_{n=1}^{\infty} c_{n+p} |a_{n+p}| \le 1 \tag{5.2}$$

and

$$c_{n+p} = \frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)} \tag{5.3}$$

then $f \in \text{k}-TUCV_p(\lambda,\alpha,\beta,\gamma,\delta)$. Furthermore,

$$\text{Re}\left\{ \frac{f(z)}{f_m(z)} \right\} > 1 - \frac{1}{c_{p+m}}, \tag{5.4}$$

$$\text{Re}\left\{ \frac{f_m(z)}{f(z)} \right\} > \frac{c_{p+m}}{1+c_{p+m}}, \tag{5.5}$$

$$\text{Re}\left\{ \frac{f'(z)}{f'_m(z)} \right\} > 1 - \frac{p+m}{c_{p+m}}. \tag{5.6}$$

Proof: For the coefficient c_{n+p} given by (5.3), it is not difficult to verify that,

$c_{n+p+1} > c_{n+p} > 1, \quad n \ge 1.$

Therefore we have

$$\sum_{n=1}^{m-1}\left|a_{n+p}\right| + c_{p+m}\sum_{n=m}^{\infty}\left|a_{n+p}\right| \le \sum_{n=1}^{\infty} c_{n+p}\left|a_{n+p}\right| \le 1. \tag{5.7}$$

By use (5.3) and by setting

$$w_1(z) = c_{p+m}\left\{\frac{f(z)}{f_m(z)} - \left(1 - \frac{1}{c_{p+m}}\right)\right\}$$

$$= 1 + \frac{c_{p+m}\sum_{n=m}^{\infty} a_{n+p} z^n}{1 + \sum_{n=1}^{m-1} a_{n+p} z^n},$$

On using (5.7), we find that for $z \in U$,

$$\left|\frac{w_1(z)-1}{w_1(z)+1}\right| \le \left|\frac{c_{p+m}\sum_{n=m}^{\infty} a_{n+p} z^n}{2 + 2\sum_{n=1}^{m-1} a_{n+p} z^n + c_{p+m}\sum_{n=m}^{\infty} a_{n+p} z^n}\right|$$

$$\le \frac{c_{p+m}\sum_{n=m}^{\infty}\left|a_{n+p}\right|}{2 - 2\sum_{n=1}^{m-1}\left|a_{n+p}\right| - c_{p+m}\sum_{n=m}^{\infty}\left|a_{n+p}\right|} \le 1,$$

which readily yields the assertion (5.4) of Theorem 5.1. Similarly, if we take

$$w_2(z) = \left(1 + c_{p+m}\right)\left\{\frac{f_m(z)}{f(z)} - \frac{c_{p+m}}{1 + c_{p+m}}\right\}$$

$$= 1 - \frac{\left(c_{p+m} + 1\right)\sum_{n=m}^{\infty} a_{n+p} z^n}{1 + \sum_{n=1}^{m-1} a_{n+p} z^n},$$

and making use of (5.3), we can deduce that

$$\left|\frac{w_2(z)-1}{w_2(z)+1}\right| \le \left|\frac{-(c_{p+m}+1)\sum_{n=m}^{\infty} a_{n+p}z^n}{2+2\sum_{n=1}^{m-1} a_{n+p}z^n-(c_{p+m}+1)\sum_{n=m}^{\infty} a_{n+p}z^n}\right|$$

$$\le \frac{(c_{p+m}-1)\sum_{n=m}^{\infty}|a_{n+p}|}{2-2\sum_{n=1}^{m-1}|a_{n+p}|-(c_{p+m}-1)\sum_{n=m}^{\infty}|a_{n+p}|} \le 1,$$

which proves (5.5). Consider,

$$w_3(z)=\frac{c_{p+m}}{p+m}\left\{\frac{f'(z)}{f_m'(z)}-\left(1-\frac{p+m}{c_{p+m}}\right)\right\}$$

$$=1+\frac{\frac{c_{p+m}}{p+m}\sum_{n=m}^{\infty}\left(\frac{p+n}{p}\right)a_{n+p}z^n}{1+\sum_{n=1}^{m-1}\left(\frac{p+n}{p}\right)a_{n+p}z^n},$$

we get,

$$\left|\frac{w_3(z)-1}{w_3(z)+1}\right| \le \left|\frac{\frac{c_{p+m}}{p+m}\sum_{n=m}^{\infty}\left(\frac{p+n}{p}\right)a_{n+p}z^n}{2+2\sum_{n=1}^{m-1}\left(\frac{p+n}{p}\right)a_{n+p}z^n+\frac{c_{p+m}}{p+m}\sum_{n=m}^{\infty}\left(\frac{p+n}{p}\right)a_{n+p}z^n}\right|$$

$$\le \frac{\frac{c_{p+m}}{p+m}\sum_{n=m}^{\infty}\left(\frac{p+n}{p}\right)|a_{n+p}|}{2-2\sum_{n=1}^{m-1}\left(\frac{p+n}{p}\right)|a_{n+p}|-\frac{c_{p+m}}{p+m}\sum_{n=m}^{\infty}|a_{n+p}|} \le 1,$$

if

$$\frac{c_{p+m}}{p+m}\sum_{n=m}^{\infty}\left(\frac{p+n}{p}\right)|a_{n+p}| \le 2-2\sum_{n=1}^{m-1}\left(\frac{p+n}{p}\right)|a_{n+p}|-\frac{c_{p+m}}{p+m}\sum_{n=m}^{\infty}|a_{n+p}|$$

that is

$$\sum_{n=1}^{m-1}\left(\frac{p+n}{p}\right)|a_{n+p}|+\frac{c_{p+m}}{p+m}\sum_{n=m}^{\infty}\left(\frac{p+n}{p}\right)|a_{n+p}| \le 1. \tag{5.8}$$

Since the left hand side of (5.8) is bounded above by $\sum_{n=1}^{\infty} c_{n+p}|a_{n+p}|$ if

$$\sum_{n=1}^{m-1}\left(\frac{p+n}{p}\right)|a_{n+p}|+\frac{c_{p+m}}{p+m}\sum_{n=m}^{\infty}\left(\frac{p+n}{p}\right)|a_{n+p}| \le \sum_{n=1}^{\infty} c_{n+p}|a_{n+p}|.$$

then

$$\sum_{n=1}^{m-1}\left(c_{n+p}-\frac{p+n}{p}\right)\left|a_{n+p}\right|+\sum_{n=m}^{\infty}\left(c_{n+p}-\frac{c_{p+m}}{p+m}\frac{p+n}{p}\right)\left|a_{n+p}\right|\geq 0, \tag{5.9}$$

which is true. This proves the assertion (5.6).

6. CONVOLUTION RESULTS

Let the functions $f_i(z)(i=1,2)$ be defined by

$$f_i(z)=z^p-\sum_{n=1}^{\infty}a_{n+p,i}z^{n+p}. \tag{6.1}$$

The Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1*f_2)(z)=z^p-\sum_{n=1}^{\infty}a_{n+p,1}a_{n+p,2}z^{n+p}, \qquad (a_{n+p,1},\ a_{n+p,2}\geq 0).$$

Theorem 6.1: Let the functions $f_i(z)(i=1,2)$ defined by (6.1) be in the class $\mathrm{k}-TUCV_p(\lambda,\alpha,\beta,\gamma,\delta)$.. Then $(f_1*f_2)(z)\in \mathrm{k}-TUCV_p(\lambda,\alpha,\beta,\gamma,\delta)$, where

$$\xi=1-\frac{p(1-\delta)^2(1+k)}{(p+\lambda)\left[(1+k)+p(1-\delta)\right]^2\phi(1,\alpha,\beta,\gamma)-p^3(1-\delta)^2}, \tag{6.2}$$

where $\phi(n,\alpha,\beta,\gamma)$ is given by (2.2).

Proof. By the technique, used earlier by Schild and Silverman [15], we need to find the largest ξ such that

$$\sum_{n=1}^{\infty}\frac{(p+n\lambda)\left[n(1+k)+p(1-\xi)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\xi)}a_{n+p,1}a_{n+p,2}\leq 1, \qquad (0\leq\xi<1)$$

for $f_i\in \mathrm{k}-TUCV_p(\lambda,\alpha,\beta,\gamma,\delta)(i=1,2)$, where ξ is defined by (6.2). On the otherhand, from (2.1) and the Cauchy's-Schwarz inequality that

$$\sum_{n=1}^{\infty}\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\sqrt{a_{n+p,1}a_{n+p,2}}\leq 1. \tag{6.3}$$

Thus we need to find the largest ξ such that

$$\sum_{n=1}^{\infty}\frac{(p+n\lambda)\left[n(1+k)+p(1-\xi)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\xi)}a_{n+p,1}a_{n+p,2}$$

$$\leq \sum_{n=1}^{\infty}\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\sqrt{a_{n+p,1}a_{n+p,2}}$$

Equivalently that,

$$\sqrt{a_{n+p,1}a_{n+p,2}} \le \frac{(1-\xi)\left[n(1+k)+p(1-\delta)\right]}{(1-\delta)\left[n(1+k)+p(1-\xi)\right]}, \qquad (n\ge 1).$$

Hence by making use of the inequality (6.3), it is sufficient to prove that

$$\frac{p^2(1-\delta)}{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)} \le \frac{(1-\xi)\left[n(1+k)+p(1-\delta)\right]}{(1-\delta)\left[n(1+k)+p(1-\xi)\right]},$$

which yields

$$\xi=\chi(n)=1-\frac{np^2(1-\delta)^2(1+k)}{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]^2\phi(n,\alpha,\beta,\gamma)-p^3(1-\delta)^2}. \tag{6.4}$$

For $\chi(n)$ is an increasing function of n $(n\ge 1)$ and letting $n=1$ in (6.4), we have

$$\xi=\chi(1)=1-\frac{p^2(1-\delta)^2(1+k)}{(p+\lambda)\left[(1+k)+p(1-\delta)\right]^2\phi(1,\alpha,\beta,\gamma)-p^3(1-\delta)^2}$$

which completes the proof of Theorem 6.1.

Theorem 6.2: Let the functions $f_i(z)(i=1,2)$ defined by (6.1) be in the class $\mathrm{k}-TUCV_p(\lambda,\alpha,\beta,\gamma,\delta)$. Then the function $h(z)$ defined by

$$h(z)=z^p-\sum_{n=1}^{\infty}\left(a_{n+p,1}^2+a_{n+p,2}^2\right)z^{n+p}$$

is in the class $\mathrm{k}-TUCV_p(\lambda,\alpha,\beta,\gamma,\delta)$ where

$$\xi=1-\frac{2p^2(1-\delta)^2(1+k)}{(p+\lambda)\left[(1+k)+p(1-\delta)\right]^2\phi(1,\alpha,\beta,\gamma)-2p^3(1-\delta)^2}, \tag{6.5}$$

where $\phi(n,\alpha,\beta,\gamma)$ is given by (2.2).

Proof. By virtue of Theorem 2.2, we obtain

$$\sum_{n=1}^{\infty}\left[\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\right]^2 a_{n+p,1}^2 \tag{6.6}$$

$$\le \sum_{n=1}^{\infty}\left[\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\, a_{n+p,1}\right]^2 \le 1$$

and

$$\sum_{n=1}^{\infty}\left[\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\right]^2 a_{n+p,2}^2$$

$$\le \sum_{n=1}^{\infty}\left[\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\, a_{n+p,2}\right]^2 \le 1. \tag{6.7}$$

It follows from (6.6) and (6.7) that

$$\sum_{n=1}^{\infty}\frac{1}{2}\left[\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\right]^2\left(a_{n+p,1}^2+a_{n+p,2}^2\right)\leq 1. \qquad (6.8)$$

Therefore we need to find the largest ξ such that

$$\frac{(p+n\lambda)\left[n(1+k)+p(1-\xi)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\xi)} \leq \frac{1}{2}\left[\frac{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]\phi(n,\alpha,\beta,\gamma)}{p^2(1-\delta)}\right]^2. \qquad (n\geq 1) \qquad (6.9)$$

That is that

$$\xi=\chi(n)=1-\frac{2p^2(1-\delta)^2(1+k)}{(p+n\lambda)\left[n(1+k)+p(1-\delta)\right]^2\phi(n,\alpha,\beta,\gamma)-2p^3(1-\delta)^2}. \qquad (6.10)$$

For $\chi(n)$ is an increasing function of n $(n\geq 1)$ and letting $n=1$ in (6.10), we have

$$\xi=\chi(1)=1-\frac{2p^2(1-\delta)^2(1+k)}{(p+\lambda)\left[(1+k)+p(1-\delta)\right]^2\phi(1,\alpha,\beta,\gamma)-2p^3(1-\delta)^2}$$

which completes the proof of this Theorem.

Remark 6.1: If we put $p=1,\ \beta=\delta$ and $\alpha=\beta$ our results are coincide with the function class $\mathrm{k}-UCV(\lambda,\ \gamma,\beta,\ \delta)$ which is investigated by [16].

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