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Quenching Behavior of Solutions in Coupled Reaction-Diffusion Systems with Singular Nonlinearities

ZeJian Cui [1] and XiaoPing Wang[2]

[1,]College of Mathematic and Information, China West Normal University, Nanchong, 637009,China
[2] College of Mathematic and Information, China West Normal University, Nanchong, 637009,China

ABSTRACT

we study a class of Reaction-Diffusion Systems with singular Nonlinearities. At first, we presents the definition of quenching, Then, in the initial boundary conditions, we study the solution of equation of quenching, quenching point set, and time derivative blows up at quenching time by using the maximum principle. Finally, we get a quenching rate by using a comparison lemma.

Keywords: Reaction-Diffusion Systems ; quenching; maximum principle

1. INTRODUCTION

In this paper, we study quenching phenomena for the following parabolic system:

$$u_t = u_{xx} + (1-u)^{-m} + (1-v)^{-p}, \qquad 0 < x < 1, 0 < t < T. \tag{1}$$
$$v_t = v_{xx} + (1-u)^{-q} + (1-v)^{-n}, \qquad 0 < x < 1, 0 < t < T. \tag{2}$$

With boundary conditions

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$$u_x(0,t) = v_x(0,t) = 0, \qquad 0 < t < T. \tag{3}$$

$$u_x(1,t) = -v^{\alpha}(1,t), \quad v_x(1,t) = -u^{\beta}(1,t) \qquad 0 < t < T. \tag{4}$$

and initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \qquad 0 \le x \le 1. \tag{5}$$

Where $m,n \ge 0$, $p,q > 0$, $\alpha,\beta > 0$, u_0 and v_0 are positive smooth functions satisfying:

$$0 < u_0 < 1, 0 < v_0 < 1; \; u_0', v_0' \le 0 \;. \tag{6}$$

For in the equations (1)-(2), their singular in the form of the original item are: $(1-u)^{-m} + (1-v)^{-p}$, $(1-u)^{-q} + (1-v)^{-n}$, Obviously, the maximum of u,v cannot reach 1. So we give the following definition of quenching:

Definition 1.1: Assume that (u,v) is the solution of problem (1)-(5), if there is a limited time T, making the following condition hold:

$$\lim_{t \to T^{-1}} \sup_{0 \le x \le 1} \max\{u(x,t), v(x,t)\} = 1.$$

Then (u,v) is called the quenching solution of problem (1)-(5), which T is the quenching time.

Throughout this paper, we assume that the initial functions (u_0, v_0) satisfies the following inequalities

$$u_0'' + (1-u_0)^{-m} + (1-v_0)^{-p} > 0, \qquad 0 \le x \le 1. \tag{7}$$

$$v_0'' + (1-u_0)^{-q} + (1-v_0)^{-n} > 0, \qquad 0 \le x \le 1. \tag{8}$$

Furthermore, In order to study quenching phenomena in (1)-(5), we need introduce some knowledge from[6].The operator:

$$L=\sum_{i,j=1}^{n} a_{i,j}(x,t)\frac{\partial^2}{\partial x_i \partial x_{ij}}+\sum_{i=1}^{n} b_i(x,t)\frac{\partial^2}{\partial x_i}-\frac{\partial}{\partial t}$$

is said to be parabolic at

$$(x,t)=(x_1,x_2,\ldots x_n,t)$$

if for fixed t the operator consisting of the first sum is elliptic at (x,t).That is, L is parabolic if there is a number $\mu>0$ such that

$$\sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \geq \mu\sum_{i=1}^{n}\xi_i^2 \quad (9)$$

For all n-tuples of real numbers $(\xi_1,\xi_2,\ldots,\xi_n)$. The operator L is uniformly parabolic in a region E_T if (9) holds with the same number $\mu>0$ for all (x,t) in E_T. Where

$$E_T=E\times(0,T), E\subset R^n; S_T=\partial E\times(0,T).$$

Theorem1.1. Assume u is a solution of the differential inequality $(L+h)[u]\geq 0$ .Provided

$$h(x,t)\leq 0.$$

(i) If $h(x,t)\equiv 0$, then $\max_{(x,t)\in E_T} u(x,t)=\max_{(x,t)\in S_T} u(x,t)$.

(ii) If $h(x,t)\geq 0$, then $\max_{(x,t)\in E_T} u(x,t)\leq\max_{(x,t)\in S_T} u(x,t)$.

For solutions of $(L+h)[u]\le 0$ there is an associated minimum principle. The result follows by application of Theorem1.1 to $(-u)$.

2. QUENCHING AND BLOW-UP OF (u_t, v_t) .

In this section, we will study set of problem (1)-(5).Then, we show that (u_t, v_t) blows up at quenching time.

Theorem2.1：Assume that (u,v) is the solution of problem (1)-(5),and (u_0, v_0) satisfies (6)-(8),then (u,v) will quenching in finite time and $x=0$ is the unique quenching point.

Proof. Since (u_0, v_0) satisfies (6)-(8), then we get $u_x, v_x > 0$ and $u_t, v_t > 0$ in $(0,1)\times(0,T)$ by the maximum principle of Theorem1.1. and thus $\max\limits_{x\in[0,1]} u(x,t) = u(0,t)$, $\max\limits_{x\in[0,1]} v(x,t) = v(0,t)$.

By the (7)-(8), we obtain the following functions satisfy:

$$w_1(t)=\int_0^1\left[(1-u_0)^{-m}+(1-v_0)^{-p}\right]dx>0,\quad w_2(t)=\int_0^1\left[(1-v_0)^{-n}+(1-u_0)^{-q}\right]dx>0.$$

Next, we give the mass functions:

$$F(t)=\int_0^1[1-u(x,t)]\,dx,\quad G(t)=\int_0^1[1-v(x,t)]\,dx.\quad 0<t<T.$$

Then

$$F'(t)=\int_0^1[-u_t(x,t)]\,dx=v^{\alpha}(1,t)-\int_0^1\left[(1-u(x,t))^{-m}+(1-v(x,t))^{-p}\right]dx\le v^{\alpha}(1,t)-w_1,$$

$$G'(t)=\int_0^1[-v_t(x,t)]\,dx=u^{\beta}(1,t)-\int_0^1\left[(1-v(x,t))^{-n}+(1-u(x))^{-q}\right]dx\le u^{\beta}(1,t)-w_2,$$

Thus, $F(t) \le F(0) + \int_0^t v^\alpha(1,t)dt - w_1 t$, and $G(t) \le G(0) + \int_0^t u^\beta(1,t)dt - w_2 t$; which means that $F(T_0) = 0$ or $G(T_0) = 0$ for some

$$T_0 = \min\left(\frac{F(0) + \int_0^t v^\alpha(1,t)dt}{w_1}, \frac{G(0) + \int_0^t u^\beta(1,t)dt}{w_2}\right), (0 < T \le T_0)$$

Thus there exist a finite time T, making:

$$\lim_{t \to T^{-1}} \sup_{0 \le x \le 1} \max\{u(x,t), v(x,t)\} = 1.$$

Then prove $x = 0$ is the only quenching point.

Define $L(x,t) = u_x + \delta x, \quad 0 < x \le \frac{1}{2}, \eta < t < T.$

Where $\eta \in (0,T)$ and δ is a positive constant to be specified later, Then

$$L_t = u_{xt} = u_{xxx} + m(1-u)^{-m-1}u_x + p(1-v)^{-p-1}v_x \le u_{xxx} = L_{xx}, 0 < x < \frac{1}{2}, \eta < t < T.$$

If δ is small enough, $L(x,\eta) < 0$ since $u_x(x,t) < 0$ in $0 < x \le 1, 0 < \eta < T$ .Furthermore, we have

$$L(x,\eta) = u_x(x,\eta) + \delta x \le 0, \quad 0 < x < \frac{1}{2}; 0 < \eta < T$$

$$L(\frac{1}{2},t) = u_x(\frac{1}{2},t) + \frac{1}{2}\delta \le 0, \quad \eta < t < T.$$

By the maximum principle of Theorem1.1, we obtain that $L(x,t) \le 0$, i.e. $u_x \le -\delta x$ .Integrating last inequality with respect to x from 0 to x , we have $u(x,t)\le u(0,t)-\frac{\delta x^2}{2}\le 1-\frac{\delta x^2}{2}$. So u does not quench in $(0,1]$. Similarly, we obtain v does not quench in $(0,1]$.

Theorem2.2： (u_t,v_t) blows up at quenching point $x=0$.

Proof. Define $J_1(x,t)=u_t-\varepsilon\left[(1-u)^{-m}+(1-v)^{-p}\right],\qquad 0\le x\le 1,\tau\le t<T.$

$$J_2(x,t)=v_t-\varepsilon\left[(1-u)^{-q}+(1-v)^{-n}\right],\qquad 0\le x\le 1,\tau\le t<T.$$

Where $\tau\in(0,T)$ and ε is a positive constant. Then, $J_1(x,t)$ and $J_2(x,t)$ satisfy

$$\begin{aligned}&(J_1)_t-(J_1)_{xx}-m(1-u)^{-m-1}J_1-p(1-v)^{-p-1}J_2=\varepsilon m(m+1)(1-u)^{-m-2}{u_x}^2+\varepsilon p(p+1)(1-v)^{-p-2}{v_x}^2\ge 0.\\&(J_2)_t-(J_2)_{xx}-q(1-u)^{-q-1}J_1-n(1-v)^{-n-1}J_2=\varepsilon q(q+1)(1-u)^{-q-2}{u_x}^2+\varepsilon n(n+1)(1-v)^{-n-2}{v_x}^2\ge 0.\end{aligned}$$

Furthermore

$$\begin{aligned}&(J_1)_x(0,t)=(J_2)_x(0,t)=0.\qquad \tau<t<T.\\&(J_1)_x(1,t)\ge 0,(J_2)_x(1,t)\ge 0.\qquad \tau<t<T.\\&J_1(x,0)=u_t(x,0)-\varepsilon\left[(1-u_0)^{-m}+(1-v_0)^{-p}\right]\end{aligned}$$

For $u_t(x,0)>0$ and $\left[(1-u_0)^{-m}+(1-v_0)^{-p}\right]$ is bounded, so if ε is small enough, we have $J_1(x,0)\ge 0$. Similarly, we have $J_2(x,0)\ge 0,(x,t)\in[0,1]\times(\tau,T)$.

Therefore, we arrive at the desired $J_1,J_2\ge 0$ by the comparison principle of Theorem1.1. i.e.

$$\begin{aligned}&u_t(0,t)\ge\varepsilon\left[(1-u(0,t))^{-m}+(1-v(0,t))^{-p}\right]\\&v_t(1,t)\ge\varepsilon\left[(1-u(0,t))^{-q}+(1-v(0,t))^{-n}\right]\end{aligned}$$

Let $t \to T^{-1}$ we have

$$\lim_{t \to T^{-1}} \max\left[u_t(0,t), v_t(0,t)\right] = \infty .$$

3. A QUENCHING CRITERION AND A QUENCHING RATE

In this section, we will gives a quenching criterion and a quenching rate. At first, we introduce a comparison.

Lemma3.1 i) If $u_0(x) \geq v_0(x), 0 \leq x \leq 1$ and $m \geq q, p \geq n; v^{\alpha}(1,t) \leq u^{\beta}(1,t)$, then

$$u(x,t) \geq v(x,t) \text{ for all } 0 \leq x \leq 1; 0 < t < T .$$

ii) If $u_0(x) \leq v_0(x), 0 \leq x \leq 1$ and $m \leq q, p \leq n; v^{\alpha}(1,t) \geq u^{\beta}(1,t)$, then $u(x,t) \leq v(x,t)$ for all $0 \leq x \leq 1; 0 < t < T$.

Proof i). Define $M(x,t) = u - v$ for all $0 \leq x \leq 1; 0 \leq t < T$ .Then $M(x,t)$ satisfies

$$\begin{aligned} M_t - M_{xx} &= (1-u)^{-m} + (1-v)^{-p} - (1-u)^{-q} - (1-v)^{-n} \\ &= (1-u)^{-m} - (1-u)^{-q} + (1-v)^{-p} - (1-v)^{-n} \\ &\geq 0 \end{aligned}$$

Further $M(x,0) \geq 0$, since $u_0(x) \geq v_0(x)$ for $0 \leq x \leq 1$.Furthermore,

$$M_x(0,t) = 0; M_x(1,t) = u^{\beta}(1,t) - v^{\alpha}(1,t) \geq 0 \quad 0 < t < T.$$

By the maximum principle of Theorem1.1, we obtain that $M(x,t) \geq 0$ for all $0 \leq x \leq 1; 0 < t < T$. i.e. $u(x,t) \geq v(x,t)$ for all $0 \leq x \leq 1; 0 < t < T$.

ii). Similarly, we can obtain $u(x,t) \le v(x,t)$ for all $0 \le x \le 1; 0 < t < T$.

Corollary3.1 From study of the problem (1)-(5), we get

$$\text{if } \lim_{t\to T^-} v(0,t) = 1, \text{ then } \lim_{t\to T^-} u_t(0,t) = \infty,$$
$$\text{if } \lim_{t\to T^-} u(0,t) = 1, \text{ then } \lim_{t\to T^-} v_t(0,t) = \infty.$$

Thus, from Theorem2.2 and lemma3.1, we get

a) u_t blows up at quenching point $x = 0$,since $u_0(x) \ge v_0(x), 0 \le x \le 1$ and $m \ge q,\ p \ge n$;

$v^\alpha(1,t) \le u^\beta(1,t)$ .Further, we have

$$u_t(0,t) \ge \varepsilon\left[(1-u(0,t))^{-m} + (1-v(0,t))^{-p}\right] \ge \varepsilon\left[(1-u(0,t))^{-m} + (1-u(0,t))^{-p}\right]$$

So, integrating for t from t to T we get $u(0,t) \le 1 - C_1(T-t)^{1/(m+1)} - C_2(T-t)^{1/(p+1)}$

Where

$$C_1 = (\varepsilon(m+1))^{1/(m+1)} \text{ and } C_2 = (\varepsilon(p+1))^{1/(p+1)}.$$

b) v_t blows up at quenching point $x = 0$,since $u_0(x) \le v_0(x), 0 \le x \le 1$ and $m \le q,\ p \le n$;

$v^\alpha(1,t) \ge u^\beta(1,t)$ .Further, we have

$$v_t(0,t) \ge \varepsilon\left[(1-u(0,t))^{-q} + (1-v(0,t))^{-n}\right] \ge \varepsilon\left[(1-v(0,t))^{-q} + (1-v(0,t))^{-n}\right]$$

So, integrating for t from t to T we get $v(0,t) \le 1 - C_3(T-t)^{1/(n+1)} - C_4(T-t)^{1/(q+1)}$

Where $C_3 = (\varepsilon(n+1))^{1/(n+1)}$ and $C_4 = (\varepsilon(q+1))^{1/(q+1)}$.

The corollary is proved.

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