Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 4 (2015), pp. 1891-1898

http://www.ripublication.com

Nonlinear Diffusion Equation of Second Order in Group Analysis

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Abstract

The paper introduces about the group analysis of a second order diffusion equation. We give the equivalence transformations, the group classification of point symmetries. Also we present nonclassical symmetries, potential symmetries. Finally we give the symmetry properties of third order dispersive equation.

Keywords: Lie Symmetry, Non classical Symmetry, Potential Symmetries, Non Classical Potential Symmetry.

1 Introduction

The investigation of nonlinear heat (or diffusion if u represents mass concentration) equations by means of symmetry methods began in 1959 with Ovsiannikov's work [10] in which the author performed the group classification of the class of equations of the form

$$\frac{du}{dt} = \frac{d}{dx}\left(f(u)\frac{du}{dx}\right) \tag{1}$$

this equation describes the stationary motion of a boundary layer of fluid over a flat plate and a vortex of incompressible fluid in a porus medium with polytropic relation between gas density and pressure. If we consider the case that the diffusion term is $f(u) = u^n$ then this equation becomes

$$\frac{du}{dt} = \frac{d}{dx}\left(u^n \frac{du}{dx}\right) \tag{2}$$

This equation is called a fast diffusion equation for $-2 < n < 0$ and a slow diffusion equation for $n > 0$. In the first case the spread of mass is much faster than in the linear case $n = 0$ and in the second case it is slower. In this section we present the known results for equation (2). We give the equivalence transformations, the group classification of point symmetries, the optimal system of one dimensional sub algebras and all possible types of invariant solutions [10]. Also we give the nonclassical symmetries [7].

2 Invariant Solutions

2.1 Equivalence transformations

We find that equation (2) admits the equivalence transformations

$$t' = c_1 t + c_2,\ x' = c_3 x + c_4,\ u' = c_1^{-1/n} c_3^{2/n} u,\ n' = n \tag{3}$$

where $c_1 c_3 \neq 0$. Furthermore, in the case for which n = - 4/3 we have additional equivalence transformations,

$$t' = c_1 t + c_2,\ x' = \frac{c_3 x + c_4}{c_5 x + c_6},\ u' = c_1^{-1/n} (c_5 x + c_6)^{-4/n} u,\ n' = n \tag{4}$$

where $c_1 \neq 0$ and $c_3 c_6 - c_4 c_5 = \pm 1$

2.2 Lie symmetries

From the definition of second order PDE admits Lie point symmetries if and only if

$\Gamma^{(2)} E\,|_{E=0} = 0$, where $\Gamma^{(2)}$ is the second prolongation of the generator

$$\Gamma = \tau(t,x,u)\partial_t + \zeta(t,x,u)\partial_x + \eta(t,x,u)\partial_u \tag{5}$$

which is given by the relation

$$\begin{aligned}\Gamma^{(2)} = \Gamma + [D_t\eta - (D_t\tau)u_t - (D_t\xi)u_x]\partial u_t + [D_x\eta - (D_x\tau)u_t - (D_x\xi)u_x]\partial u_x \\ + [D_x\eta^x - (D_x\tau)u_{xt} - (D_x\xi)u_{xx}]\partial u_{xx}.\end{aligned} \tag{6}$$

Here D_t and D_x are the total derivatives with respect to t and x respectively and η^x is the coefficient function of ∂u_x. In this case we have that

$$E = u_t - u^n u_{xx} - n u^{n-1} u_x^2 = 0 \tag{7}$$

and eqn(2) admits Lie point symmetries if and only if

$$\Gamma^{(2)}\left[u_t - u^n u_{xx} - n u^{n-1} u_x^2\right] = 0 \tag{8}$$

for $u_t = u^n u_{xx} + n u^{n-1} u_x^2$. After the elimination of u_t due to the above expression equation (8) becomes an identity in the variables u_x, u_{tx}, u_{xx}. The coefficients of different powers of these variables must be zero and these variables must be zero and these give the determining equations on the coefficients τ, ξ, η. Using the general results on point transformations between evolution equations [5] that τ = τ(t) and ξ = ξ(t, x) we take the following determining equations from the coefficients of $u_{xx},\ u_x^2,\ u_x$ and the term independent of derivatives of (8) equation respectively,

$$(\tau_t - 2\xi_x)u + n\eta = 0 \tag{9}$$

$$\eta_{uu}u^2 + n(\tau_t - 2\xi_x + \eta_u)u + n(n-1)\eta = 0 \tag{10}$$

$$(\xi_{xx} - 2\eta_{xu})u^{n+1} - 2n\eta_x u^n - \xi_t u = 0 \tag{11}$$

$$\eta_{xx}u^n - \eta_t = 0. \tag{12}$$

When we solve these equations (9-12) we observe that for the case when n is arbitrary, the symmetry Lie algebra is four dimensional and is spanned by

$$\Gamma_1 = \partial_t,\ \Gamma_2 = \partial_x,\ \Gamma_3 = 2t\partial_t + xdx,\ \Gamma_4 = \frac{nx}{2}\partial_x + u\partial_u. \qquad (13)$$

An additional Lie symmetry exists for the specific value of the parameter n = -4/3. In particular, equation (2) admits a fifth symmetry

$$\Gamma_5 = x^2\partial_x + \frac{4xu}{n}\partial_u \qquad (14)$$

The primary use of Lie symmetries is to obtain a reduction of variables. Similarity variables appear as constants of integration in the solutions of the characteristic equations.

$$\frac{dt}{\tau} = \frac{dx}{\xi} = \frac{du}{\eta} \qquad (15)$$

Reductions could be obtained from any symmetry which is an arbitrary linear combination that is,

$$a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4 + a_5\Gamma_5 \qquad (16)$$

To ensure that a minimal complete set of reductions is obtained from the Lie symmetries of equation (2), we construct so called optimal system of sub-algebras. In the case for which n is arbitrary the optimal system and the corresponding similarity reductions that transform (2) into an ODE are given by operators,

$$\langle\Gamma_2\rangle : u = \Phi(\omega), \omega = t \qquad (17)$$

$$\langle\Gamma_1 + c\Gamma_2\rangle : u = \Phi(\omega)(\omega), \omega = x - ct \qquad (18)$$

$$\langle\Gamma_3 + c\Gamma_4\rangle : u = t^{\frac{c}{2}}\Phi(\omega), \omega = \begin{cases} x & if\ nc+2=0, \\ t^{-\frac{1}{2}}x^{\frac{2}{nc+2}} & nc+2\neq 0 \end{cases} \qquad (19)$$

$$\langle\Gamma_4 + c\Gamma_1\rangle : u = x^{\frac{2}{n}}\Phi(\omega), \omega = \begin{cases} e^{t}x^{\frac{-2c}{n}} & if\ n\neq 0, \\ e^{\frac{t}{c}}\Phi(\omega),\ \omega = x & n=0 \end{cases} \qquad (20)$$

$$\langle\Gamma_4 + c\Gamma_2 - \frac{n}{2}\Gamma_3\rangle : u = t^{-\frac{1}{n}}\Phi(\omega), \omega = \begin{cases} x + \frac{c}{n}\ell n t & if\ n\neq 0 \\ e^{\frac{x}{c}}\Phi(\omega), \omega = t & n=0 \end{cases} \qquad (21)$$

In the special case n = -4/3 for which a fifth symmetry exists, we obtain the following additional reductions that corresponds to the additional sub algebras:

$$\langle \Gamma_5 + c\Gamma_2 + 2k\Gamma_3 \rangle : u = \begin{cases} \left((x+k)^2+1\right)^{\frac{2}{n}} \times \exp\left[\frac{4k}{n}\tan^{-1}(x+k)\right]\Phi(\omega), \\ \quad \omega = t\exp\left[-4k\tan^{-1}(x+k)\right] \quad if\ c-k^2=1 \\ \left((x+k)^2-1\right)^{\frac{2}{n}} \exp\left[\frac{4k}{n}\tanh^{-1}(x+k)\right]\Phi(\omega) \\ \quad \omega = t\exp\left[-4k\tanh^{-1}(x+k)\right] \quad if\ c-k^2=-1 \\ (x+k)^{\frac{4}{n}} \exp\left[\frac{4k}{n}(x+k)\right]\Phi(\omega) \\ \quad \omega = t\exp\left[\frac{-4k}{x+k}\right] \quad if\ c-k^2=0 \end{cases} \tag{22}$$

where ω is the independent ,Φ is the dependent variable of the reduced ODE, c = 0, ±1 and k ∈R.

3 Non classical Symmetries

In this case we require the invariance system of PDEs

$$u_t = (u^n u_x)_x,\ \tau(t,x,u)u_t + \xi(t,x,u)u_x = \eta(t,x,u) \tag{23}$$

under the class of infinitesimal transformations generated by

$$\Gamma = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u \tag{24}$$

This results in an over determined nonlinear system of PDE's for the determination of the coefficients τ(t, x, u), ξ(t, x, u) ,η(t, x, u)∂_u. After we choose τ = 1, the nonclassical method applied to the equation (2) gives rise to the four nonlinear determining equations

$$\xi_{uu}u - n\xi_u = 0 \tag{25}$$

$$(2\xi_{xu} - \eta_{uu})u^{n+2} - n\eta_u u^{n+1} + n\eta u^n - 2\xi\xi_u u^2 = 0 \tag{26}$$

$$(\xi_{xx} - 2\eta_{xu})u^{n+1} - 2n\eta_x u^n - (\xi_t + 2\xi\xi_x - 2\xi_u\eta)u + n\xi\eta = 0 \tag{27}$$

$$(\eta_{xx})u^{n+1} - (2\xi_x\eta + \eta_t)u + n\eta^2 = 0 \tag{28}$$

After we have solved the determining system (25) -(28) we can assure that equation (2) admits a proper nonclassical symmetry only for n = -1/2. For n ≠ -1/2 we only recover the classical symmetries. For n = -1/2 we obtain the nonclassical symmetry

$$\Gamma = \partial_t + \phi(x)\sqrt{u}\,\partial_u, \tag{29}$$

where $\phi(x)$ is a solution of ODE

$$\frac{d^2\phi}{dx^2} - \frac{1}{2}\phi^2 = 0 \tag{30}$$

A particular solution of this equation is $\phi = 12x^2$. The nonclassical operator above produces the non classical reduction

$$u = \left[\frac{1}{2}\phi(x)t + F(x)\right]^2 \tag{31}$$

which reduces (2) to ODE

$$\frac{d^2 f}{dx^2} - \frac{1}{2}\phi F = 0 \quad (32)$$

If $\phi = 12x^{-2}$, this ODE becomes an equation of Euler type with the form

$$x^2 \frac{d^2 F}{dx^2} - 6F = 0 \quad (33)$$

and solution

$$F(x) = c_1 x^{-2} + c_2 x^3 \quad (34)$$

which yields explicit solution for(2)

$$u(t,x) = \left(6t\, x^{-2} + c_1 x^{-2} + c_2 x^3\right)^2 \quad (35)$$

4 Potential Symmetries

If we introduce the potential variable v, we can write equation (2) as a system of PDEs,

$$v_x = u,\ v_t = u^n u_x \quad (36)$$

Suppose equation (36) admits a infinitesimal generator of the form

$$\Gamma = \tau(t,x,u,v)\partial_t + \xi(t,x,u,v)\partial_x + \eta(t,x,u,v)\partial_u + \zeta(t,x,u,v)\partial_v \quad (37)$$

We search for Lie point symmetries for the system (36) with the optimal goal of finding potential symmetries for equation (2). Lie symmetries of (36) induce potential symmetries for (2) if the following condition holds, $\tau_v^2 + \xi_v^2 + \eta_v^2 \neq 0$. The system (36) admits Lie symmetries if and only if $\Gamma^{(1)}[v_x - u] = 0,\ \Gamma^{(1)}[v_t - u^n u_x] = 0$ (38) where system (36) holds. Here $\Gamma^{(1)}$ is the first extension generator (37) and is given by the relation.

$$\Gamma^{(1)} = \Gamma + [D_t\eta - (D_t\tau)u_t - (D_t\xi)u_x]\partial u_t + [D_x\eta - (D_x\tau)u_t - (D_x\xi)u_x]\partial u_x$$
$$+ [D_t\zeta - (D_t\tau)v_t - (D_t\xi)v_x]\partial v_t + [D_x\zeta - (D_x\tau)v_t - (D_x\xi)v_x]\partial v_x.$$

(Here D_t and D_x are the total derivatives with respect to t and x , respectively) Eliminating D_t and D_x through substitution of (36) into (38) we obtain seven determining equations for τ, ξ, η, ζ which simplify to:

$$\tau_u = 0 \quad (39)$$

$$\tau_v u + \tau_x = 0 \quad (40)$$

$$\xi_u u - \zeta_u = 0 \quad (41)$$

$$\tau_v u^n - \xi_u = 0 \quad (42)$$

$$\eta_v u^{n+1} + \eta_x u^n + \xi_t u - \zeta_t = 0 \quad (43)$$

$$(\tau_t - \xi_x + \_\eta_u - \zeta_v)u = 0 \quad (44)$$

$$\left(\xi_v u^2 + (\xi_x - \zeta_v)u - \zeta_x\right) + \eta = 0 \quad (45)$$

Solution of the determining equations (39 -45) can be summarised as follows:

For the case that $n \neq -2$, the system (36) admits a five parameter group with infinitesimal generators.

$$\Gamma_1=\partial_t,\ \Gamma_2=\partial_x,\ \Gamma_3=2t\partial_t+x\partial_x+v\partial_v,\ \Gamma_4=\partial_v,\ \Gamma_5=x\partial_x+\frac{2u}{n}\partial_u+\left(1+\frac{2}{n}\right)v\partial_v. \quad (46)$$

When n = -2, the system (36) admits an infinite parameter group with infinitesimal generators (46)

$$\Gamma_6=-xv\partial_x+u(xu+v)\partial_u+2t\partial_v,\quad \Gamma_7=4t^2\partial_t-x\left(2t+v^2\right)\partial_x+u\left(6t+2xuv+v^2\right)\partial_v, \quad (47)$$

$$\Gamma_8 = f\partial_x-u^2 f_v\partial_u \quad (48)$$

where f = f(t,v) is the solution of the linear heat equation $f_t = f_{vv}$. The Lie symmetries (46) project into local symmetries of (2) and the Lie symmetries Γ_6, Γ_7, Γ_8 induce potential symmetries for the corresponding equation (2)

5 Nonclassical potential Symmetries

In this case we search for nonclassical symmetries for a potential system or potential equation. It is easier to search for nonclassical symmetries for the potential equation which is obtained from the associated auxiliary system of (2) given by

$$v_x = u,\ v_t = u^n u_x. \quad (49)$$

To achieve this we eliminate u from the above system we get

$$v_t = v_x^n v_{xx} \quad (50)$$

which is the potential form of equation (2).Now we take into consideration the case n = -1 for which the potential equation becomes

$$v_t = v_x^{-1} v_{xx} \quad (51)$$

Here the invariance surface condition of the form

$$\tau(t,x,v)v_t + \xi(t,x,v)v_x = \zeta(t,x,v) \quad (52)$$

and the reduction operators have the general form

$$\Gamma = \tau(t,x,v)\partial_t + \xi(t,x,v)\partial_x + \zeta(t,x,v)\partial_v \quad (53)$$

We assume that τ = 1 without loss of generality. The four determining equation coefficients ξ, ζ have the form

$$\xi_{vv} - \xi\xi_v = 0 \quad (54)$$

$$\zeta_{xx} - \zeta\zeta_x = 0 \quad (55)$$

$$\xi_t - 2\xi_{xv} + \zeta_{vv} + \xi\zeta_v - \xi_v\zeta + \xi\xi_x = 0 \quad (56)$$

$$\zeta_t - 2\zeta_{xv} + \xi_{xx} + \xi_x\zeta - \xi\zeta_x + \zeta\zeta_v = 0. \quad (57)$$

The nonclassical symmetries of the potential fast diffusion equation (51) that result from the solution of the determining equations (54- 57) are

$$\Gamma_1 = \partial_t + \xi\partial_x + f(\omega)\partial_v,\quad where\quad \omega = x+\xi t$$

$$\Gamma_2 = \partial_t f(\omega)(\partial_x + \partial_v),\quad where\ \omega = x+v$$

$$\Gamma_3 = \partial_t + \xi\partial_x + (\phi_t + \phi_x\xi)\partial_v,\quad where\ \xi = \frac{-2}{v+\phi}\ and\ \phi\in\left\{t+e^x, tf(x)\right\}$$

$$\Gamma_4 = \partial_t + \xi\partial_x - \frac{g_t + g_x\xi}{1+g^2}\partial_v,\quad where\ \xi = -2\frac{1+g\tan v}{\tan v - g}\ and\ g\in\{\tan(2t)\tanh x, \coth(2t)\cot x\}$$

$$\Gamma_5 = \partial_t + \xi\partial_x - \frac{g_t + g_x\xi}{1-g^2}\partial_v, \quad where\ \xi = -2\frac{1-g\tanh v}{\tanh v - g}\ and$$

$$g \in \left\{ \tanh(2t)\tanh x, \tanh(2t)\coth x, \coth(2t)\coth x, \frac{\left(e^{2x}\tanh(2t)+1\right)}{e^{2x}-\tanh(2t)}, \frac{2-e^{2x}-e^{-4t}}{2+e^{2x}-e^{4t}} \right\}$$

Here $\xi \in \{0,1\}$ and f is arbitrary nonconstant solution of ODE $f_{\omega\omega} = ff_\omega$ with the solution being in parametric form $\omega = \int \frac{df}{\frac{f^2}{2}+c} + c_1$ which leads to a particular solution

$$f = \begin{cases} \sqrt{2c}\left(\sqrt{c/2}(\omega - c_1)\right) & if \quad c \succ 0 \\ \sqrt{2|c|}\left[\frac{1+\exp\left(\sqrt{2|c|}(\omega - c_1)\right)}{1-\exp\left(\sqrt{2|c|}(\omega - c_1)\right)}\right] & if \quad c \prec 0 \\ \frac{2}{c_1 - \omega} & if \quad c = 0. \end{cases}$$

Conclusion

The Main goal of this paper was the investigations of symmetry properties for special classes of nonlinear evaluation PDEs. Our motivation started from the known results of the nonlinear diffusion equation.

$$\frac{du}{dt} = \frac{d}{dx}\left(u^n \frac{du}{dx}\right)$$

In this paper studied differential equations which depend upon parameters and for certain values of these parameters we obtained useful symmetry properties. One of the main goals that we had was to find patterns between the values of the parameters for which exceptional symmetries occur.

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