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Existence and Uniqueness result for certain nonlinear Boundary value problems

M.B. Okofu and C.L. Ejikeme
Department of Mathematics,
University of Nigeria, Nsukka, Nigeria.

Abstract

In this paper, we study the existence and uniqueness of solution of certain non-linear boundary value problem involving p-Laplacian. Furthermore, we show that the solution of the non-linear boundary value problem is zero of a certain maximal monotone operation. Our result extends the corresponding result of *Wei et al.* in [10].

AMS subject classification: 47H05, 47H09.
Keywords: Monotone operator, generalized p-Laplacian operator, non-linear boundary value problem, Sobolev spaces.

1. Introduction

Some significant work have been done by *Wei et al.* [1, 5, 2, 4, 3, 7, 10, 6]. on the study of nonlinear boundary value problems involving p-Laplacian operator, ∇_p. In 1995, Wei and He [2] used a perturbation result of ranges for m-accretive mappings in Calvert and Gupta [1] to obtain a sufficient condition so that the zero boundary value problem,

$$\begin{aligned} -\nabla_p u + g(x, u(x)) &= f(x), \text{ a.e in } \Omega, \\ -\frac{\partial u}{\underline{\partial n}} &= 0, \text{ a.e in } \Gamma, \end{aligned} \tag{1.1}$$

has solutions in $L^p(\Omega)$, where $2 \leq p < +\infty$. Other works from different angles have also been done on this kind of equation, [][4,6,15]. In 2004,they studied the following nonlinear elliptic boundary value problem involving the generalized p-Laplacian operator:

$$\begin{aligned} -div[(c(x) + |\nabla u|^2)^{(p-2)/2}\nabla u] + |u|^{p-2}u + g(x, u(x)) &= f(x), \text{ a.e in } \Omega \\ -\langle v, (c(x) + |\nabla u|^2)^{p-2/2}\nabla u\rangle &= 0, \text{ a.e in } \Gamma \end{aligned} \tag{1.2}$$

In Wei and Zhou [5], they proved that (1.2) has solutions in $L^2(\Omega)$, where $2 \leq p < \infty$, under certain conditions. This work was later extended in [15,16] to the problem:

$$-div[(c(x) + |\nabla u|^2)^{(p-2)/2}\nabla u] + |u|^{p-2}u + g(x, u(x)) = f(x), \text{a.e in } \Omega \quad (1.3)$$
$$-\langle v, (c(x) + |\nabla u|^2)^{p-2/2}\nabla u\rangle \in \beta_x(u(x)), \text{ a.e in } \Gamma$$

Wei and Zhou [3], showed that (1.3) has solutions in $L^S(\Omega)$, where $2 \leq p < +\infty$, and in [16], Wei proved that (1.3) has solutions in $L^s(\Omega)$, where $max(N, 2) \leq p < +\infty$. As an extension and summary of the work done in [15,16], in 2008, they used new techniques to work for the the nonlinear boundary value problem with generalized p-Laplacian operator:

$$-div[(c(x) + |\nabla u|^2)^{(p-2)/2}\nabla u] + \epsilon|u|^{q-2}u + g(x, u(x)) = f(x), \text{a.e in } \Omega \quad (1.4)$$
$$-\langle v, (c(x) + |\nabla u|^2)^{p-2/2}\nabla u\rangle \in \beta_x(u(x)), \text{ a.e in } \Gamma$$

where $0 \leq c(x) \in L^p(\Omega)$, ϵ is a nonnegative constant and v denotes the exterior normal derivative of Γ. In [7] ,Wei and Agrawal, showed that (1.4) has solutions in $L^s(\Omega)$ under certain conditions, where $2N/(N+1) < p \leq s < +\infty$ if $p \geq N$, and $1 \leq q \leq Np/(N-p)$ if $p < N$, for $N \geq 1$. Inspired by the work of Chen and Luo [8], who studied the eigenvalue problem for the generalized Capillary equations:

$$-div\Big[\Big(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u\Big] = \lambda(|u|^{q-2}u + |u|^{r-2}u), \ in \ \Omega, \quad (1.5)$$
$$u = 0, \text{a.e. on } \partial\Omega.$$

in their paper [10], Wei et al., studied the generalized Capillary equations with Neuamnn boundary conditions

$$-div\Big[\Big(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u\Big] + \lambda(|u|^{q-2}u + |u|^{r-2}u) + g(x, u(x))$$
$$= f(x), \text{a.e. in } \Omega, \quad (1.6)$$

$$-\Big\langle v, \Big(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u\Big\rangle \in \beta_x(u(x)), \text{ a.e on } \Gamma$$

by using the perturbation results of ranges for m-accretive mappings in Calvert and Gupta [1]. The techniques used were different from those in Chen and Luo [8].

Motivated by [10, 12], in this paper, we extend the work done in [10] to study the Capillary equation

$$-div\Big[\Big(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u\Big] + \lambda(|u|^{q-2}u + |u|^{r-2}u + |u|^{s-2}u) \quad (1.7)$$
$$+ g(x, u(x), \nabla u(x)) = f(x), \text{a.e. in}\Omega,$$

$$-\Big\langle v, \Big(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u\Big\rangle \in \beta_x(u(x)), \text{ a.e in } \Gamma$$

This equation generalizes the Capillarity problem considered in [10]. Here, we replace the nonlinear term $g(x, u(x))$ by the term $g(x, u(x), \nabla u(x))$ which is more general. We will use some perturbation results of the ranges for maximal monotone operators by Pascali and Shurlan [10] to prove that (1.7) has a unique solution in $W^{1,p}(\Omega)$ and later show that this unique solution is the zero point of a suitably defined maximal monotone operator.

2. Preliminaries

Next we give some notations, definitions and lemmas which will be relevant in the sequel. Let X be a real Banach space with a strictly convex dual space X^*. Let " $\hookrightarrow$ " and "$w - lim$" denote strong and weak convergence respectively. For any subset G of X, let $intG$ denote its interior and $\bar{G}$ its closure. Let "$X \hookrightarrow\hookrightarrow Y$" denote that space X is embedded compactly in space Y and "$X \hookrightarrow Y$" denote that space X is embedded continuously in space Y. $(.\,,.)$ denotes the generalized duality pairing X and X^*. A single-valued mapping, $T : D(T) = X \to X^*$ is said to be hemicontinuous on X if $w - \lim_{t\to 0} T(x + ty) = Tx$, for any $x, y \in X$.

Let $A : X \to 2^X$ be a given multi-valued mapping, we say that A is boundedly-inversely compact if for any pair of bounded subsets G and G' of X, the subset $G \bigcap A^{-1}(G')$ is relatively compact in X.

The mapping $A : X \to 2^X$ is said to be accretive if $(v_1 - v_2, J(u_1 - u_2)) \geq 0$, for any $u_i \in D(A)$ and $v_i \in Au_i$; $i = 1, 2$.

The accretive mapping A is said to be m-accretive if $R(I + \mu A) = X$, for some $\mu > 0$. Let $B : X \to 2^{X^*}$ be a given multi-valued mapping, the graph of B, $G(B)$ is defined by $G(B) = \{[u, w] \mid u \in D(B), w \in Bu\}$. $B : X \to 2^{X^*}$ is said to be monotone [11] if $G(B)$ is a monotone subset of $X \times X^*$ in the sense that

$$(u_1 - u_2, w_1 - w_2) \geq 0, \text{ for any } [u_i, w_i] \in G(B); i = 1, 2. \tag{2.1}$$

The monotone operator B is said to be maximal monotone if $G(B)$ is maximal among all monotone subsets of $X \times X^*$ in the sense of inclusion the mapping B is said to be strictly monotone if the equality in (2.1) implies that $u_1 = u_2$. The mapping B is said to be coercive if $\lim_{n\to+\infty} ((x_n, x_n^*)/||x_n||) = \infty$ for all $[x_n, x_n^*] \in G(B)$ such that $\lim_{n\to+\infty} ||x_n|| = +\infty$.

Function Φ is called a proper convex function of X if Φ is defined from X to $(-\infty, +\infty]$, not identically $+\infty$ such that $\Phi[(1 - \lambda)x + \lambda y] \leq (1 - \lambda)\Phi(x) + \lambda\Phi(y)$ whenever $x, y \in X$and$0 \leq \lambda \leq 1$.

Function $\Phi : X \to (-\infty, +\infty]$ is said to be lower semicontinuous on X if,

$$\liminf_{y\to x} \Phi(y) \geq \Phi(x), \text{ for any } x \in X$$

Given a proper convex function Φ on X and a point $x \in X$, we denote by $\partial\Phi(x)$ the set of all $x^* \in X^*$ such that $\Phi(x) \leq \Phi(y) + (x - y, x^*)$, for every $y \in X$. Such elements x^* are called subgradients of Φ at x, and $\partial\Phi(x)$ is called the subdifferential of Φ at x.

A point $x \in D(B)$ is said to be a zero point of B if $0 \in Ax$, and we denote by $B^{-1}(0) = \{x \in X : 0 \in Ax\}$ the set of zero points of A.

Let J denote the duality mapping from X to 2^{X^*} defined by

$$J(x) = f \in X^* : (x, f) = ||x||.||f||, ||f|| = ||x||, \ x \in \ X \tag{2.2}$$

Definition 2.1. The duality mapping $J : X \to 2^{X^*}$ is said to be satisfying condition (I) if there exists a function $\eta : X \to [0, +\infty)$ such that

$$||Ju - J\upsilon|| \leq \eta(u - \upsilon), \text{ for all } u, \ \upsilon \in X. \tag{2.3}$$

Definition 2.2. Let $A : X \to 2^X$ be an accretive mapping and $J : X \to X^*$ be a duality mapping. We say that A satisfies condition $(*)$ if, for any $f \in R(A)$ and $a \in R(A)$ and $a \in D(A)$, there exists a constant $C(a, F)$ such that

$$(\upsilon - f, J(u - a)) \geq C(a, f), \text{ for any } u \in D(A), \ \upsilon \in Au. \tag{2.4}$$

Lemma 2.3. (Li and Guo) Let Ω be a bounded conical domain in R^N. Then we have the following results;

1. If $mp > N$ then $W^{m,p}(\Omega) \hookrightarrow C_B(\Omega)$; if $mp < N$ and $q = Np/(N - mp)$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$; if $mp = N$, and $p > 1$, then for $1 \leq q < +\infty$, $W^{m,p}(\Omega) \to L^q(\Omega)$

2. If $mp > N$ then $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$; if $0 < mp < N$ and $q_0 = Np/(N - mp)$, then $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, $1 \leq q < \infty$;

Lemma 2.4. (Adams [17]) Let Ω be a bounded conical domain in $\Re^N$, if $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow C_B(\Omega)$; if $mp < N$ and $q = Np/(N - mp)$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$; if $mp = N$ and $p > 1$, then for $1 \leq q < +\infty$, $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$.

Lemma 2.5. (Pascali and Sburlan [11]) If $B : X \to 2^{X^*}$ is everywhere defined, monotone and hemicontinuous operator, then B is maximal monotone.

Lemma 2.6. (Pascali and Sburlan [11]) If $B : X \to 2^{X^*}$ is maximal monotone and coercive, then $R(B) = X^*$.

Lemma 2.7. (Pascali and Sburlan [11]) If $\Phi : X \to (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function, then $\partial\Phi$ is maxiamal monotone from $X \; to \; X^*$.

Lemma 2.8. [11] If B_1 and B_2 are two maximal monotone operators in X such that $(int D(B_1)) \bigcap D(B_2) \neq \emptyset$, then $B_1 + B_2$ is maximal monotone.

Lemma 2.9. (Calvert and Gupta [1]) Let $X = L^p(\Omega)$ and Ω be a bounded domain in $\Re^N$. For $2 \le p < +\infty$, the duality mapping $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ defined by $J_P u = |u|^{p-1} sgn\, u ||u||_p^{2-p}$, for $u \in L^P(\Omega)$, satisfies condition (2.4); for $2N/(N+1) < p \le 2$ and $N \ge 1$, the duality mapping $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ defined by $J_P u = |u|^{p-1} sgn\, u$, for $u \in L^P(\Omega)$, satisfies condition (2.4), where $(1/p) + (1/p') = 1$.

Lemma 2.10. (Calvert and Gupta [1]) Let Ω be a bounded domain in $\Re^N$ and $g : \omega \times \Re^N \to \Re$ be a function satisfying Caratheodory's conditions such that

1. $g(x, .)$ is monotonicity increasing on $\Re$;

2. the mapping $u \in L^p(\Omega) \to g(x, u(x)) \in L^P(\Omega)$ is well defined, where $2N/(N+1) < p < +\infty$ and $N \ge 1$.

Let $J_p : L^p(\Omega) \to L^{p'}(\Omega)$, $(1/p) + (1/p') = 1$ be the duality mapping defined by

$$J_p u = \begin{cases} |u|^{p-1} sgn\, u, \quad if\ \dfrac{2N}{N+1} < p \le 2 \\ \\ |u|^{p-1} sgn\, u ||u||_p^{2-p}, \quad if\ 2 \le p < +\infty, \end{cases} \tag{2.5}$$

for $u \in L^p(\Omega)$. The mapping $B : L^p(\Omega) \to L^p(\Omega)$ defined by $(Bu)(x) = g(x, u(x))$, for any $x \in \Omega$ satisfies Condition (2.4).

3. Main result

3.1. Assumptions of (1.7)

We assume that $2N/(N+1) < p < +\infty$, $1 \le q, r, s < +\infty$ if $p \ge N$, and $1b \le q, r, s \le Np/(N-p)$ if p<N, where $N \ge 1$. We use $||.||_{p'}$, $||.||_{q'}$, $||.||_{r'}$, $||.||_{s'} and ||.||_{1,p,\Omega}$ to denote the norms in $L^p(\Omega)$, $L^q(\Omega)$, $L^r(\Omega)$, $L^s(\Omega)$ and $W^{1,p}(\Omega)$. Let $(1/p)+(1/p') = 1$, $(1/q) + (1/q') = 1$, $(1/r) + (1/r') = 1$ and $(1/s) + (1/s') = 1$.

In (1.7), Ω is a bounded conical domain of a Euclidean space $\Re^N$ with its boundary $\Gamma \in C^1$, $(c.f.$[4]). Let $|.|$ denote the norm in $\Re^N$, $\langle ., . \rangle$ the Euclidean inner-product and υ the exterior normal derivative of Γ. λ is a nonnegative constant and υ denotes the exterior normal derivative of Γ.

3.2. Existence and Uniqueness of Solution of (1.7)

Lemma 3.1. D efine the mapping $B_{p,q,r,s} : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by

$$\begin{aligned}(v, B_{p,q,r,s}) &= \int_\Omega \Big\langle (1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}})|\nabla u|^{p-2}\nabla u, \nabla v \Big\rangle dx \\ &+\lambda \int_\Omega |u(x)|^{q-2}u(x)v(x)dx \\ &+\lambda \int_\Omega |u(x)|^{r-2}u(x)v(x)dx + \lambda \int_\Omega |u(x)|^{s-2}u(x)v(x)dx \quad (3.1)\end{aligned}$$

for all $u, v \in W^{1,p}(\Omega)$. Then $B_{p,q,r,s}$ is everywhere defined, strictly monotone, hemicontinuous and coercive.

The proof of the above lemma will be done in four steps:

Proof. Step 1: $B_{p,q,r,s}$ is everywhere defined. From lemma 1.3, $W^{1,p}(\Omega) \hookrightarrow C_B(\Omega)$, when $p > N$. Also, $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, $W^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$, when $p \le N$. Hence, for all

$$v \in W^{1,p}(\Omega),\ ||v||_q \le k_1||v||_{1,p,\Omega},\ ||v||_r \le k_2||v||_{1,p,\Omega},\ ||v||_s \le k_3||v||_{1,p,\Omega},$$

where k_1, k_2, k_3 are positive constants. For $u, v \in W^{1,p}(\Omega)$,

$$\begin{aligned}|(v, B_{p,q,r,s}| &\le 2\int_\Omega |\nabla u|^{p-1}|\nabla v|dx + \lambda \int_\Omega |u|^{q-1}|v|dx \\ &+\lambda \int_\Omega |u|^{r-1}|v|dx + \lambda \int_\Omega |u|^{s-1}|v|dx \\ &\le 2|||\nabla u|||_p^{p/p'}|||\nabla v|||_p + \lambda||v||_q||u||_q^{q/q'} \\ &+\lambda||v||_r||u||_r^{r/r'} + \lambda||v||_s||u||_s^{s/s'} \\ &\le 2||u||_{1,p,\Omega}^{p/p'}||v||_{1,p,\Omega} + k_1'\lambda||v||_{1,p,\Omega}||u||_{1,p,\Omega}^{q/q'} + k_2'\lambda||v||_{1,p,\Omega}||u||_{1,p,\Omega}^{r/r'} \\ &+k_3'\lambda||v||_{1,p,\Omega}||u||_{1,p,\Omega}^{s/s'} \quad (3.2)\end{aligned}$$

where k_1', k_2' and k_3' are positive constants. Thus $B_{p,q,r,s}$ is everywhere defined.

Step 2: $B_{p,q,r,s}$ is strictly monotone. For $u, v \in W^{1,p}(\Omega)$, we have

$$
\begin{aligned}
&|(u-v, B_{p,q,r,s}u - B_{p,q,r,s}v)| \\
&\quad = \int_\Omega \Big\langle \Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^{p-2}\nabla u - \Big(1 + \frac{|\nabla v|^p}{\sqrt{1+|\nabla v|^{2p}}}\Big)|\nabla v|^{p-2}\nabla v, \nabla u - \nabla v\Big\rangle dx \\
&\qquad + \lambda \int_\Omega (|u|^{q-2}u - |v|^{q-2}v)(u-v)dx + \lambda \int_\Omega (|u|^{r-2}u - |v|^{r-2}v)(u-v)dx \\
&\qquad + \lambda \int_\Omega (|u|^{s-2}u - |v|^{s-2}v)(u-v)dx \\
&\quad = \int_\Omega \Big\{\Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^p - \Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^{p-2}\nabla u \nabla v \\
&\qquad - \Big(1 + \frac{|\nabla v|^p}{\sqrt{1+|\nabla v|^{2p}}}\Big)|\nabla v|^{p-2}\nabla u \nabla v + \Big(1 + \frac{|\nabla v|^p}{\sqrt{1+|\nabla v|^{2p}}}\Big)|\nabla v|^p\Big\} \\
&\qquad + \lambda \int_\Omega (|u|^{q-2}u - |v|^{q-2}v)(u-v)dx + \lambda \int_\Omega (|u|^{r-2}u - |v|^{r-2}v)(u-v)dx \\
&\qquad + \lambda \int_\Omega (|u|^{s-2}u - |v|^{s-2}v)(u-v)dx \\
&\quad \geq \int_\Omega \Big\{\Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^{p-1} + \Big(1 + \frac{|\nabla v|^p}{\sqrt{1+|\nabla v|^{2p}}}\Big)|\nabla v|^{p-1}\Big\}(|\nabla u| - |\nabla v|)dx \\
&\qquad + \lambda \int_\Omega (|u|^{q-1} - |v|^{q-1})(|u| - |v|)dx + \lambda \int_\Omega (|u|^{r-1} - |v|^{r-1})(|u| - |v|)dx \\
&\qquad + \lambda \int_\Omega (|u|^{s-1} - |v|^{s-1})(|u| - |v|)dx
\end{aligned} \tag{3.3}
$$

If we let $h(t) = (1 + \frac{t}{\sqrt{1+t^2}})t^{p-1}/p$, for $t \geq 0$. Then

$$
h'(t) = \frac{t^{(p-1)/p}}{(1+t^2)^{3/2}} + t^{-(1/p)}\Big(1 + \frac{t}{\sqrt{1+t^2}}\Big)\frac{p-1}{p} \geq 0, \tag{3.4}
$$

since $t \geq 0$. And, $h'(t) = 0$ if and only if $t = 0$. Then $h(t)$ is strictly monotone. Thus we can say that $B_{p,q,r,s}$ is strictly monotone.

Step 3: $B_{p,q,r,s}$ is hemicontinuous.
Here we need to show that, for any $u, v, w \in W^{1,p}(\omega)$ and $t \in [0, 1]$, $(w, B_{p,q,r,s}(u + tv) - B_{p,q,r,s}u) \to 0$ as $t \to 0$.

By Lebesque's dominated convergence theorem, it follows that

$$\begin{aligned}
0 &\leq \lim_{t\to 0}|(w, B_{p,q,r,s}(u+tv) - B_{p,q,r,s}u)| \\
&\leq \int_\Omega \lim_{t\to 0}\Big|\Big(1+\frac{|\nabla u + t\nabla v|^P}{\sqrt{1+|\nabla u + t\nabla v|^2 p}}\Big)|\nabla u + t\nabla v|^{p-2}(\nabla u - t\nabla v) \\
&- \Big(1+\frac{|\nabla u|^P}{\sqrt{1+|\nabla u|^2 p}}\Big)|\nabla u|^{p-2}\nabla u\Big||\nabla w|dx \\
&+ \lambda\int_\Omega \lim_{t\to 0}\Big||u+tv|^{q-2}(u+tv) - |u|^{q-2}u\Big||w|dx \\
&+ \lambda\int_\Omega \lim_{t\to 0}\Big||u+tv|^{r-2}(u+tv) - |u|^{r-2}u\Big||w|dx \\
&+ \lambda\int_\Omega \lim_{t\to 0}\Big||u+tv|^{s-2}(u+tv) - |u|^{s-2}u\Big||w|dx \\
&= 0 \qquad (3.5)
\end{aligned}$$

Therefore $B_{p,q,r,s}$ is hemicontinuous.

Step 4: $B_{p,q,r,s}$ is Coercive.
For $u \in W^{1,p}(\Omega)$, Lemma 2.4 implies that $||u||_{1,p,\Omega} \to \infty$ is equivalent to $||u - (1/meas(\Omega))\int_\Omega udx||_{1,p,\Omega} \to \infty$ and hence we have the following result:

$$\begin{aligned}
\frac{(u, B_{p,q,r,s}u)}{||u||_{1,p,\Omega}} &= \frac{\int_\Omega\Big(1+\Big(|\nabla u|^p/\sqrt{1+|\nabla u|^{2p}}\Big)\Big)|\nabla u|^p dx}{||u||_{1,p,\Omega}} + \lambda\frac{\int_\Omega |u|^q dx}{||u||_{1,p,\Omega}} \\
&+\lambda\frac{\int_\Omega |u|^r dx}{||u||_{1,p,\Omega}} + \lambda\frac{\int_\Omega |u|^s dx}{||u||_{1,p,\Omega}} \\
&= \frac{\int_\Omega\Big(|\nabla u|^p + \sqrt{1+|\nabla u|^{2p}}\Big)dx - \int_\Omega\Big(\sqrt{1/|\nabla u|^{2p}}\Big)dx}{||u||_{1,p,\Omega}} \\
&+\lambda\frac{\int_\Omega |u|^q dx}{||u||_{1,p,\Omega}} + \lambda\frac{\int_\Omega |u|^r dx}{||u||_{1,p,\Omega}} + \lambda\frac{\int_\Omega |u|^s dx}{||u||_{1,p,\Omega}} \\
&\leq \frac{2\int_\Omega |\nabla u|^p dx - \int_\Omega\Big(1/\sqrt{1+|\nabla u|^{2p}}\Big)dx}{||u||_{1,p,\Omega}} + \lambda\frac{\int_\Omega |u|^q dx}{||u||_{1,p,\Omega}} \\
&+\lambda\frac{\int_\Omega |u|^r dx}{||u||_{1,p,\Omega}} + \lambda\frac{\int_\Omega |u|^s dx}{||u||_{1,p,\Omega}} \to +\infty, \qquad (3.6)
\end{aligned}$$

as $||u||_{1,p,\Omega} \to +\infty$, which implies that $B_{p,q,r,s}$ is coercive. This completes the proof. ∎

Remark 3.2. This lemma is an important result for later use.

Lemma 3.3. (Wei and Agrawal [7]) The mapping $\Phi_p : W^{1,p}(\Omega) \to R$ defined by

$$\Phi_p(u) = \int_\Gamma \varphi_x(u|_\Gamma(x))d\Gamma(x), \tag{3.7}$$

for any $u \in W^{1,p}$ is proper, convex, and lower semicontinuous on $W^{1,p}(\Omega)$. (lemma 2.7) implies that $\partial\Phi$, is maximal monotone.

Definition 3.4. Define a mapping $A_p : L^p(\Omega) \to 2^{L^p(\Omega)}$ as follows:

$$\begin{aligned} D(A_p) &= \{u \in L^p(\Omega) \mid \text{there exist an } f \in L^p(\Omega), \\ &\quad \text{such that } f \in B_{p,q,r,s}u + \partial\Phi_p(u)\} \qquad (3.8) \\ &\text{for } u \in D(A_p),\ let\ A_p u = \{f \in L^p(\Omega), \\ &\quad \text{such that } f \in B_{p,q,r,s}u + \partial\Phi_p(u)\} \qquad (3.9) \end{aligned}$$

Definition 3.5. The mapping $A_p : L^p(\Omega) \to 2^{L^p(\Omega)}$ is m-accretive.

Proof.

1. A_p is accretive.

   (a) Case 1:
   If $p \geq 2$, the duality mapping $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ is defined by $J_p u = |u|^{p-1}sgnu||u||_p^{2-p}$ for $u \in L^p(\Omega)$. We need to prove that for any $u_i \in D(A_p)$ and $v_i \in A_p u_i$, $i = 1, 2$,

   $$(v_1 - v_2, J_p(u_1 - u_2)) \geq 0 \tag{3.10}$$

   we are left to prove that

   $$\begin{aligned} &(|u_1 - u_2|^{p-1}sgn(u_1 - u_2)||u_1 - u_2||_p^{2-p}, B_{p,q,r,s}u_1 - B_{p,q,r,s}u_2) \geq 0, \\ &(|u_1 - u_2|^{p-1}sgn(u_1 - u_2)||u_1 - u_2||_p^{2-p}, \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) \geq 0, \end{aligned} \tag{3.11}$$

   are available. Take for a constant $k > 0$, $\chi_k : R \to R$ is defined by $\chi_k(t) = |(t \bigwedge k) \bigvee (-k)|^{p-1}sgnt.|$ Then χ_k is monotone, Lipschitz with $\chi_k(0) = 0$ and χ_k' is continuous except at finitely many points on R. This gives that

   $$\begin{aligned} &(|u_1 - u_2|^{p-1}sgn(u_1 - u_2)||u_1 - u_2||_p^{2-p}, \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) \\ &= \lim_{k \to +\infty} ||u_1 - u_2||_p^{2-p}(\chi_k(u_1 - u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) \geq 0, \end{aligned} \tag{3.12}$$

Also

$$
\begin{aligned}
&(|u_1 - u_2|^{p-1} sgn(u_1 - u_2)||u_1 - u_2||_p^{2-p}, B_{p,q,r,s}u_1 - B_{p,q,r,s}u_2) \\
&\quad = ||u_1 - u_2||_p^{2-p} \\
&\quad \times \lim_{k\to+\infty} \int_\Omega \Big\langle \Big(1 + \frac{|\nabla u_1|^p}{\sqrt{1+|\nabla u_1|^{2p}}}\Big)|\nabla u_1|^{p-2}\nabla u_1 \\
&\quad -\Big(1 + \frac{|\nabla u_2|^p}{\sqrt{1+|\nabla u_2|^{2p}}}\Big)|\nabla u_2|^{p-2}\nabla u_2, \nabla u_1 - \nabla u_2\Big\rangle \\
&\quad \times \chi_k'(u_1 - u_2)dx + \lambda||u_1 - u_2||_p^{2-p} \int_\Omega (|u_1|^{q-2}u_1 \\
&\quad -|u_2|^{q-2}u_2)|u_1 - u_2|^{p-1} sgn(u_1 - u_2)dx \\
&\quad +\lambda||u_1 - u_2||_p^{2-p} \int_\Omega (|u_1|^{r-2}u_1 \\
&\quad -|u_2|^{r-2}u_2)|u_1 - u_2|^{p-1} sgn(u_1 - u_2)dx \\
&\quad +\lambda||u_1 - u_2||_p^{2-p} \int_\Omega (|u_1|^{s-2}u_1 \\
&\quad -|u_2|^{s-2}u_2)|u_1 - u_2|^{p-1} sgn(u_1 - u_2)dx \geq 0 \qquad (3.13)
\end{aligned}
$$

The last inequality is available since χ_k is monotone and $\chi_k(0) = 0$

(b) case 2
If $2N/(N+1) < p < 2$, the duality mapping $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ is defined by $J_p(u) = |u|^{p-1} sgnu$, for $u \in L^p(\Omega)$. It then suffices to prove that for any $u_i \in D(A_p)$ and $v_i \in A_p u_i$, $i = 1, 2$

$$(v_1 - v_2, J_p(u_1 - u_2)) \geq 0 \qquad (3.14)$$

To this, we define the function $\chi_k : R \to R$ by

$$\chi_n(t) = \begin{cases} |t|^{p-1}\, sgn\, t, \quad if\ |t| \geq \dfrac{1}{n}, \\ \\ (\dfrac{1}{n})^{p-2}\, t, \quad if\ |t| \leq \dfrac{1}{n} \end{cases} \qquad (3.15)$$

■

Lemma 3.6. Define the mapping $F : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by

$$(v, Fu(x)) = \int_\Omega g(x, u(x), \nabla(x))v(x)dx, \qquad (3.16)$$

for all $u, v \in W^{1,p}(\Omega)$: then F is everywhere defined, monotone and hemicontinuous on $W^{1,p}(\Omega)$.

Proof. Again, we split the proof into four steps:

Step 1
For $u \in W^{1,p}(\Omega), x \rightarrow g(x, u(x), \nabla(x))$ is measurable on Ω From the facts that $u(x), \partial u/\partial x_i \in L^P(\Omega), i = 1, 2, \ldots, N$, we know that $x \rightarrow (u(x), \partial u/\partial x_1, \partial u/\partial x_N)$ is measurable on Ω. In addition to the fact that g satisfies Carathéodory's conditions, we know that $x \rightarrow g(x, u(x), \nabla(x))$ is measurable on Ω.

Step 2
F is everywhere defined. From 2.4 we know that $W^{1,p}(\Omega) \hookrightarrow C_B(\Omega)$, when $p > N$ and $W^{1,p}(\Omega) \hookrightarrow L^{p'}(\Omega)$ when $p + N$. Therefore, for all $w \in W^{1,p}(\Omega)$,

$$||w||_{p'} \leq ||w||_{1,p,\Omega'} \tag{3.17}$$

where $k > 0$ is a constant. When $p < N$ from 2.4 we know that $W^{1,p}(\Omega) \hookrightarrow L^{Np/(N-p)}(\Omega)$. Since $2N/(N+1) < p < +\infty$, then $Np/(N-p) > p'$ and $L^{Np/(N-p)}(\Omega) \hookrightarrow L^{p'}(\Omega)$, which implies that (3.17) is true. Now, for $u, v \in W^{1,p}(\Omega)$, from (3.17),

$$\begin{aligned} |(v, Fu)| &\leq \int_\Omega |h_1(x)||v|dx + b\int_\Omega |u|^{p-1}|v|dx + b\sum_{i=1}^{N}\int_\Omega |\frac{\partial u}{\partial x_i}|^{p-1}|v|dx \\ &\leq ||h_1(x)||_p||v||_{p'} + b||u||_p^{p/p'}||v||_p + b\sum_{i=1}^{N}||\frac{\partial u}{\partial x_i}||_p^{p/p'}||v||_p \qquad (3.18) \\ &\leq ||h_1(x)||_p||v||_{1,p,\Omega} + b(N+1)||u||_{1,p,\Omega}^{p/p'}||v||_{1,p,\Omega'} \end{aligned}$$

Which implies that F is everywhere defined.

Step 3
F is monotone. Since $g(x, r_1, \ldots, r_{N+1})$ is monotone with respect to r_1, then F is monotone.

Step 4. F is hemicontinuous Here we need to show that, for any $u, v, w \in W^{1,p}$ and $t \in [0, 1]$, $(w, F(u + tv) - Fu) \rightarrow 0$ as $t \rightarrow 0$. Noting that g is measurable on Ω and g satisfies the Catatheédory's conditions, by using the Lebesques's dominated Convergence theorem, we have

$$\begin{aligned} 0 &\leq \lim_{t\rightarrow} |(w, F(u + tv) - Fu)| \\ &\leq \int_\Omega \lim_{t\rightarrow 0} |g(x, u(x) + tv(x), \nabla u + t\nabla v) - g(x, u, \nabla u)||w|dx = 0 \quad (3.19) \end{aligned}$$

And hence F is hemicontinuous. This completes the proof. ■

By lemma 2.5, we have that: $B_{p,q,r,s}$ and F are maximal monotone operators. And by lemma 2.8 we obtain the following result.

Lemma 3.7. $B_{p,q,r,s} + F + \partial\Phi_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is *maximal monotone*.

Lemma 3.8. $R(B_{p,q,r,s} + F + \partial\Phi_p) = (W^{1,p}(\Omega))^*$.

Proof. From Lemma 3.5 [9] we know that for all $u \in W^{1,p}(\Omega)$,

$$(u, \partial\Phi_p(u)) = \int_\Gamma \beta_x(u|_\Gamma(x))u|_\Gamma(x)d\Gamma(x) \tag{3.20}$$

Since $0 \in \beta_x(0)$, then for all $u \in W^{1,p}(\Omega)$; $(u, \partial\Phi_p(u)) \geq 0$. Since F is monotone, $(u, Fu - F0) \geq 0$. Furthermore, by (3.17) we have

$$\begin{aligned}\frac{|(u, F0)|}{||u||_{1,p,\Omega}} &\leq \frac{\int_\Omega |h_1(x)||u(x)|dx}{||u||_{1,p,\Omega}} \\ &\leq ||h_1(x)||_p \frac{||u||'_p}{||u||_{1,p,\Omega}} \leq k||h_1(x)||_p < +\infty.\end{aligned} \tag{3.21}$$

Then by lemma 3.1 we have;

$$\begin{aligned}&(u, B_{p,q,r,s}u + Fu + \partial\Phi_p(u))||u||_{1,p,\Omega} \\ &\geq \frac{(u, B_{p,q,r,s}u)}{||u||_{1,p,\Omega}} + \frac{(u, F0)}{||u||_{1,p,\Omega}} \to +\infty; \text{ as } ||u||_{1,p,\Omega} \to +\infty\end{aligned} \tag{3.22}$$

which implies that $B_{p,q,r,s}u + Fu + \partial\Phi_p$ is coercive. In view of lemmas 3.7 and 2.5 we get the result;

$$R(B_{p,q,r,s} + F + \partial\Phi_p) = (W^{1,p}(\Omega))^*$$

The proof is complete. ■

Theorem 3.9. For $f \in L^{p'}(\Omega)$, the nonlinear boundary value problem (1.7) has a unique solution $u \in W^{1,p}(\Omega)$.

Then χ_n is monotone, Lipschitz with $\chi_n(0) = 0$ and χ_n' is continuous except at finitely many points on R. so that $(\chi_n(u_1 - u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) \geq 0$. Then for $u_i \in D(A_p)$ and $v_i \in A_p u_i$, $i = 1, 2$, we have

$$\begin{aligned}(v_1 - v_2, J_p(u_1 - u_2)) &= (|u_1 - u_2|^{p-1}sgn(u_1 - u_2), B_{p,q,r,s}u_1 - B_{p,q,r,s}u_2) \\ &\quad +(|u_1 - u_2|^{p-1}sgn(u_1 - u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) \\ &= (|u_1 - u_2|^{p-1}sgn(u_1 - u_2), B_{p,q,r,s}u_1 - B_{p,q,r,s}u_2) \\ &\quad + \lim_{n\to\infty}(\chi_n(u_1 - u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) \geq o\end{aligned} \tag{3.23}$$

Step 2 $R(l + \mu A_p) = L^P(\Omega)$, for every $\mu > 0$.
Define the mapping $I_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by $I_p u = u$ and $(v, I_p u)(W^{1,p}(\Omega))^* \times W^{1,p}(\Omega) = (u, v)L^P(\Omega)$ for $u, v \in W^{1,p}(\Omega)$, where $\langle ., .\rangle_{L^2(\Omega)}$ denotes the inner product of $L^2(\Omega)$. Then I_p is monotone [7].

For any $\mu > 0$, let the mapping $T_\mu : W^{1,p}(\Omega) \to 2^{(W^{1,p}(\Omega))^*}$ be defined by $T\mu u = I_p u + \mu B_{p,q,r,s} u \mu\partial\Phi_p(u)$, for $u \in W^{1,p}(\Omega)$. Then similar to that in [7], by lemmas 2.4, 2.6, 2.7 and 2.5 we see that T_μ is maximal monotone and coercive, so that $R(T\mu) = (W^{1,p}(\Omega))^*$, for any $\mu > 0$. Therefore, for any $f \in L^P(\Omega)$, there exists $u \in W^{1,p}(\Omega)$, such that

$$f = T_\mu u = u + \mu B_{p,q,r,s} u \mu\partial\Phi_p(u) \tag{3.24}$$

From the definition of A_p, it follows that $R(I + \mu A_p) = L^p(\Omega)$, for all $\mu > 0$. This ends the proof. ■

Lemma 3.10. The mapping $A_p : L^p(\Omega) \to 2L^p(\Omega)$, has a compact resolvent for $2N/(N+1) < p < 2$ and $N \geq 1$.

Proof. Since A_p is m=accretive , we need to show that if $u + \mu A_p = f(\mu > 0)$ and if $\{f\}$ is bounded in $L^p(\Omega)$. Now let χ_n, be as defined in (3.13) and define the function $\zeta_n : R \to R$ by

$$\zeta_n(t) = \begin{cases} |t|^{2-(2/p)}\, sgn\ t, \quad if\ |t| \geq \dfrac{1}{n}, \\ \left(\dfrac{1}{n}\right)^{1-(2/p)} t, \quad if\ |t| \leq \dfrac{1}{n} \end{cases} \tag{3.25}$$

Noting that $\chi_n'(t) = (p-1)\times(p'/2)^p \times (\zeta_n'(t))^p$, for $|t| \geq 1/n$, where $(1/p)+1/p') = 1$ and $\chi_n(t) = (\zeta_n(t))^p$, for $|t| \leq 1/n$. We know that $(\chi_n, \partial\Gamma_p(\Omega)(u)) \geq 0$ for $u \in W^{1,p}(\Omega)$ since χ_n is monotone, Lipschitz with $\chi_n(0) = 0$ and χ_n' is continuous except at finitely many points on R.

$$\begin{aligned}
(|u|^{p-1} sgn\ u, A_p u) &= \lim_{n\to\infty}(\chi_n(u), A_p u) \geq \lim_{n\to\infty}(\chi_n(u), B_{p,q,r,s}u) \\
&= \lim_{n\to} \int_\omega \Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^p \chi_n'(u)dx \\
&\quad +\lambda \lim_{n\to\infty}\int_\Omega |u|^{q-2}u\chi_n(u)dx \\
&\quad +\lambda \lim_{n\to\infty}\int_\Omega |u|^{r-2}u\chi_n(u)dx + \lambda \lim_{n\to\infty}\int_\Omega |u|^{s-2}u\chi_n(u)dx \\
&\geq \lim_{n\to}\int_\omega |\nabla u|^p \chi_n'(u)dx \\
&\geq const. \lim_{n\to\infty}\int_\Omega |grad|(\zeta(u))|^p dx \\
&\geq const. \int_\Omega |grad(|u|^{2-(2/p)} sgn\ u)|^p dx
\end{aligned} \tag{3.26}$$

From $f = u + \mu A_p u$, we have;

$$\begin{aligned}\|f\|_p \left|\left| |u|^{2-(2/p)} Sgn\ u \right|\right|_{p^2/2(p-1)}^{p^2/2(p-1)p'} &\geq (|u|^{p-1} sgn\ u, f) \\ &= (|u|^{p-1} sgn\ u, u) + \mu(|u|^{p-1} sgn\ u, A_p u) \\ &\geq \left|\left| |u|^{2-(2/p)} Sgn\ u \right|\right|_{p^2/2(p-1)}^{p^2/2(p-1)p'} \\ &\quad + \mu.const. \left|\left| grad|u|^{2-(2/p)} sgn\ u \right|\right|^{p'_p}\end{aligned}$$

which gives that

$$\left|\left| |u|^{2-(2/p)} Sgn\ u \right|\right|_p^{p/2(p-1)} \leq \left|\left| |u|^{2-(2/p)} Sgn\ u \right|\right|_{p^2/2(p-1)}^{p/2(p-1)} ||f||_p \leq const. \quad (3.27)$$

In view of the fact that $p < \dfrac{P^2}{2(p-1)}$ when $2N(N+1) < p < 2$ for $N \geq 1$. Again from (3.23), $||grad(|u|^{2-(2/p)} sgn\ u)||_p \leq const.$ Hence, {f} bounded in $L^P(\Omega)$ implies that $\{|u|^{2-(2/p)} sgn\ u\}$ is bounded in $W^{1,p}(\Omega)$. We notice that $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{p^2/2(p-1)}(\Omega)$ when $N \geq 2$ and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$ when $N = 1$ by lemma (2.4), therefore $\{|u|^{2-(2/p)} sgn\ u)\}$ is relatively compact in $L^{p^2/2(p-1)}(\Omega)$. This gives that $\{u\}$ is relatively compact in $L^P(\Omega)$ since the Nemytskii mapping $u \in L^{p^2/2(p-1)}(\Omega) \to |u|^{p^2/2(p-1)} sgn\ u \in L^p(\Omega)$ is continuous. This completes the proof. ■

Remark 3.11. Since $\Phi_p(u + \alpha) = \Gamma_p(u)$, for any $u \in W^{1,p}(\Omega)$ and $\alpha \in C_0^\infty(\Omega 0)$, we have $f \in A_p u$ implies that $f = B_{p,q,r,s} u$ in the sense of distributions.

Proposition 3.12. For $f \in L^p(\Omega)$, there exists $u \in L^p(\Omega)$ such that $f \in A_p u$, then u is the unique solution of (1.7).

Proof. First we show that

$$\begin{aligned}&-div\Big[\Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u\Big] + \lambda(|u|^{q-2}u + |u|^{r-2}u + |u|^{s-2}u) \\ &= f(x), \text{ a.e.} x\ in \Omega,\end{aligned} \quad (3.28)$$

is available. $f \in A_p u$ implies that $f = B_{p,q,r,s} u + \partial\Phi_p(u)$. For all $\varphi \in C_0^\infty(\Omega)$, by remark (3.12), we have;

$$\begin{aligned}
(\varphi, f) &= (\varphi, B_{p,q,r,s}u + \partial\Phi_p(u)) \\
&= (\varphi, B_{p,q,r,s}u) = \int_\Omega \Big\langle\Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u, \nabla\varphi\Big\rangle dx \\
&+ \lambda\int_\Omega |u|^{q-2}u\varphi dx + \lambda\int_\Omega |u|^{r-2}u\varphi dx + \lambda\int_\Omega |u|^{s-2}u\varphi dx \\
&= \int_\Omega -div\Big[\Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u\Big]\varphi dx \\
&+ \lambda\int_\Omega |u|^{q-2}u\varphi dx + \lambda\int_\Omega |u|^{r-2}u\varphi dx + \lambda\int_\Omega |u|^{s-2}u\varphi dx
\end{aligned}$$

which implies that (3.25) is true.
Secondly, we show that

$$-\Big\langle v, \Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u, \Big\rangle \in \beta_x(u(x)), \; a.e\ x \ \in \Gamma \tag{3.29}$$

this will be proved under the condition that $|\beta_x(u)| \le a|u|^{p/p'} + b(x)$, where $b(x) \in L^{p'}(\Gamma)$ and $a \in R$. From (3.25), $f \in A_p u$ implies that

$$\begin{aligned}
f(x) = -div\Big[\Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u\Big] \\
+ \lambda|u|^{q-2}udx + \lambda|u|^{r-2}u + \lambda|u|^{s-2}u \in L^p(\Omega).
\end{aligned}$$

By Green's Formula, we have that for any $v \in W^{1,p}(\Omega)$,

$$\begin{aligned}
&\int_\Gamma \Big\langle v, \Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u, \Big\rangle v|_\Gamma d\Gamma(x) \\
&= \int_\Omega div\Big[\Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u\Big] v d(x) \\
&+ \int_\Omega \Big\langle\Big(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u, \nabla v\Big\rangle d(x)
\end{aligned}$$

Then $-\Big\langle v, \Big(1+|\nabla u|^p/\sqrt{1+|\nabla u|^{2p}}\Big)|\nabla u|^{(p-2)}\nabla u, \Big\rangle \in W^{-(1/p)p'}(\Gamma) = (W^{1/p.p'}(\Gamma))^*$, where $W^{1/p.p'}(\Gamma)$ is the space of traces of $W^{1,p}(\Omega)$. Let the mapping $B : L^p(\Gamma) \to L^{P'}(\Gamma)$ be defined by $Bu = g(x)$, for any $u \in L^p(\Gamma)$, where $g(x) = \beta_x(u(x))$ a.e on Γ. Clearly, $B = \partial\Psi$ where

$$\Psi(u) = \int_\Gamma \varphi_x(u(x))d\Gamma(x)$$

is a proper, convex and lower-semicontinuous function on $L^p(\Gamma)$. Now define the mapping $K : W^{1,p}(\Omega) \to L^p(\Gamma)$ by

$$K(v) = v|_\Gamma \text{ for any } v \in W^{1,p}(\Omega)$$

Then $K * BK : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is maximal monotone since both K, B are continuous. Finally, for any $u, v \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}\Psi(Kv) - \Psi(Ku) &= \int_\Gamma [\varphi_x(v|_\Gamma(x)) - \varphi_x(u|_\Gamma(x))]d\Gamma(x) \\ &\geq \int_\Gamma \beta_x(u|_\Gamma(x)(v|_\Gamma(x)) - u|_\Gamma(x))d\Gamma(x) \\ &= (BKu, Kv - Ku) = (K * BKu, v - u).\end{aligned} \tag{3.30}$$

Hence we get that $K * BK \subset \partial\Phi_p$ and so $K * BK = \partial\Phi_p$. Therefore, we have (3.25) is true. Next we show that u is unique.

If $f \in A_p u$ and $f \in A_p v$, where $u, v \in D(A_p)$. Then

$$0 \leq (u - v, B_{p,q,r,s}u - B_{p,q,r,s}v) = (u - v, \partial\Phi(v) - \partial\Phi(u)) \leq 0 \tag{3.31}$$

Bp, q, r, s being strictly monotone and $\partial\Phi_p$ maximal monotone, implies that $u(x) = v(x)$. This completes the proof. ■

Remark 3.13. If $\beta_x \equiv 0$ for any $x \in \Gamma$ then $\partial\Phi_p(u) \equiv 0$, for all $u \in W^{1,p}(\Omega)$.

Proposition 3.14. If $\beta_x \equiv 0$ for any $x \in \Gamma$ then $\{f \in L^P(\Omega) | \int_\Omega f dx = 0\} \subset R(A_p)$.

Proof. In view of Lemmas 2.4, 2.5 and 3.1 we note that $R(B_{p,q,r,s}) = (W^{1,p}(\Omega))^*$. Note also that for any $f \in L^P(\omega)$ with $\int_\Omega f dx = 0$, the linear function $u \in W^{1,p}(\omega) \to \int_\Omega f u dx$ is an element of $(W^{1,p}(\Omega))^*$. So there exists a $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}\int_\Omega f v dx &= \int_\Omega \Big\langle\Big(1 + \frac{|\nabla u|^p}{\sqrt{1 + |\nabla u|^{2p}}}\Big)|\nabla u|^{(p-2)}\nabla u, \nabla v\Big\rangle dx \\ &+ \lambda \int_\Omega |u|^{q-2}uv dx + \lambda \int_\Omega |u|^{r-2}uv dx + \lambda \int_\Omega |u|^{s-2}uv dx\end{aligned} \tag{3.32}$$

for any $v \in W^{1,p}(\Omega)$. Therefore, $f = A_P u$ in view of Remark 3.12. This completes the proof. ■

Definition 3.15. (see[1,7]) For $t \in R_t, x \in \Gamma$, let $\beta_x^0(t) = \beta_x(t)$ be the element with least absolute value if $\beta_x(t) \neq \emptyset$ and $\beta_x^0(t) = \pm\infty$, where $t > 0$ or $t < 0$ respectively, in case $\beta_x(t) = \emptyset$. Finally, let $\beta_x(t) = \lim_{t\to\pm\infty} \beta_x^0(t)$ (in the extended sense) for $x \in \Gamma.\beta_x(t)$ define measurable functions on Γ, in view of our assumptions on β_x.

Proposition 3.16. Let $f \in L^p(\Omega)$ such that

$$\int_\Gamma \beta_-(x)d\Gamma(x) < \int_\Gamma f dx < \int_\Gamma \beta_+(x)d\Gamma(x) \tag{3.33}$$

Then $f \in R(A_p)$.

Proof. Let $f \in L^p(\Omega)$ and satisfy (3.31), by proposition 3.5, there exists $u_n \in L^p(\Omega)$ such that, for each $n \geq 1$, $f = (1/n)u_n + A_p u_n$. In the same reason as that in[1], we only need to prove that$||u_n||_p <$ const for all $n \geq 1$.

Indeed suppose to the contrary that $1 \leq ||u_n||_p \to \infty$, as $n \to \infty$. Let $u_n = u_n/||u_n||_p$. Let $\psi : R \to R$ be defined by $\psi(t) = |t|^p$, $\partial\psi : R \to R$ be its sub-differential and for $\mu > 0$, $\partial\psi_\mu : R \to R$ denote the Yosida- approximation of $\partial\psi$. Let $\theta_\mu : R \to R$ denote the indefinite integral of $[(\partial\psi_\mu)']^{1/p}$ with $\theta_\mu = 0$ so that $(\theta_p')^p = (\partial\psi_\mu)'$. In view of [1] we have

$$(\partial\psi_\mu(v_n), \partial\Phi_p(u_n)) \geq \int_\Gamma \beta_x((1+\mu\partial\psi)^{-1}(u_n|_\Gamma(x))) \times \partial\psi_\mu(v_n|_\Gamma(x))d\Gamma(x) \geq 0 \tag{3.34}$$

Now multiplying the equation $f = (1/n)u_n + A_p u_n$ by $\partial\psi_\mu(v_n)$, we get that

(3.35)

■

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