Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 4 (2015), pp. 1859-1871

http://www.ripublication.com

Design and Analysis of Lagrangian Decomposition Model

[a] Subramani R and [b] Vijayalakshmi C

[a,b] *SAS, Mathematics Division, VIT University, Chennai, India – 600127*
subramanivit@gmail.com vijusesha2002@yahoo.co.in

Abstract:

This paper mainly deals with design of Lagrangian decomposition algorithm. Decomposition algorithms are analyzed with respect to various parameters and conditions. Dual decomposition, and more generally Lagrangian relaxation, is a classical method for combinatorial optimization; it has recently been applied to several inference problems. Lagrangian Relaxation (LR) technique decomposes the optimization problem into subproblems; Lagrangian subproblems give the optimal solutions for the optimization problem. The aim of this paper is to identify the control and uncontrolled parameters of the various decomposition techniques which are framed as equality, inequality constraints. Particularly the Lagrangian multipliers added in the objective function of the Lagrangian problem which is acting as "penalty factors", based on the parameters of the system. It is compared with the other decomposition techniques such as primal decomposition, dual decomposition. A main theme of this paper is that Lagrangian relaxation is obviously applied in conjunction with a wide class of combinatorial algorithms, allowing inference in models that provides the appropriate optimal solutions.

Keywords: Optimization, Nonlinear Programming Problem (NLPP), primal decomposition, dual decomposition, Lagrangian Multiplier, Lagrangian Relaxation,

Mathematics Subject Classification (2010) 90C11 . 90C25 . 90C30

1.0 Introduction:

Mathematical optimization including numerical techniques has its origin in operations research developed in 1940s. During the World War II, the military administration in

England assigned a team of specialists to study the strategic and tactical problems of air and land defense. Many of these problems were of execution type. The objective was to determine the most effectual utilization of the limited military wealth. Optimization is a Mathematical discipline that optimizes the objective functions, subject to the set of constraints. George Dantzig used mathematical techniques for generating "programs" (timetables, scheduling and training) for military force application. Since then, his "Linear Programming" techniques and are applied to a wide multiplicity of problems, from scheduling of the production facilities, to yield managing in airlines.

The Large number of real-world problems in optimization involves several conflicting objectives which should be measured all together, so-called vector-optimization problems. The process of solution these problems are threefold, based on decision-making method, methods to treat nonlinear constraints and optimization algorithms to optimize the objective function. In these problems, the optimal solution is found, or most professional, way of using restricted resources to achieve the objective based on the situation. This may be optimizing the project. For the given problem can be formulated by a mathematical sketch called a mathematical model to represent appropriate situation.

A linear programming (LP) model consists of a linear objective function subject to the linear constraints in the form of decision variables. If a model consists of a nonlinear objective function subject to the nonlinear constraints, then it is called nonlinear programming model. Dantzig was developed an algorithm so called Simplex Algorithm which is used to solve linear programming problems (LPP). Using the same technique can in two or higher dimensions. The most outstanding operations research technique is designed for LP models. Other techniques include Integer Programming Problems (IPP) (variables are assumed integer variables), Dynamic Programming Problems (DPP) (the subproblems are decomposed from the original model), Network Programming (a network can be framed according given situation) and Non Linear Programming Problem (NLPP) (in which the functions of the models are nonlinear).

1.1 Review of Literature:

In the past research, Bissan (2015) et al present a Lagrangian decomposition approach that exploits the structure of the problem leading to smaller problems that are solved independently. Daniel E. Marthaler (2013) et al gives an overview of Mathematical Methods for Numerical optimization. An optimization problem was solved using excel solver method which doesn't require multifarious knowledge in mathematical concepts by Leslie Chandrakantha (2012) et al. Researcher Hakan Terelius (2010) et al gives a solution of decentralilized optimization problem using primal and dual algorithms. Haldun (2015) et al proposes an exact method, based on Generalized Benders Decomposition, to select the best *M* features during induction. Vidar Gunnerud (2011) et al discussed Decomposition strategies for petroleum production problem which exploit the structure of a problem and divide it into smaller tractable subproblems. Alexander (2011) et al proposes a traditional manner for combinatorial optimization so-called dual decomposition, have been functioned to a number of implication problems in accepted language processing. Matthew Galati (2010) was developed a theoretical outline for a convex hull of sufficient solutions of an IPP together a variety of decomposition-based methods.

2.0 Mathematical model formulation:

2.1 Parameters:

- $f_0 : R^n \rightarrow R$: objective function
- $x=(x_1,.....,x_n)$: design variables (unknowns of the problem, they must be linearly independent vectors in R^n)
- $g_i : R^n \rightarrow R$: (i=1,…,m) : inequality constraints
- $h_i : R^n \rightarrow R$: (j=1,…,r): equality constraints
- f(x), g_j (x) and h_j(x) : scalar functions of the real column vector X
- λ = Lagrangian Multiplier
- L[x , λ] = Lagrangian function

Minimize f(x), $x=[x_1,x_2,\ldots,x_n]^T \in R^n$

Subject to the constraints,

$$g_j(x) \leq 0, \qquad j=1, 2, 3, m \tag{1}$$

$$h_j(x) = 0, \qquad j=1, 2, 3,\ldots, r$$

The continuous components x_i of $x=[x_1, x_2, \ldots, x_n]^T$ are called the (pattern) variables. The optimal vector x that solves problem (1) is denoted by x* with subsequent optimal function value f(x*). If no constraints are precises then the proposed problem is called an *unconstrained* minimization problem. Mathematical Optimization is often also called NLP, Mathematical Programming or Numerical Optimization which gives the best solutions to precisely defined problems. There are two types of solutions over there for optimization problems. In the first case solutions corresponds that to minimum energy configurations of universal structures, from molecules to suspension bridges, and thus of significance to Science and Engineering. In the next consists of economic significance to Society and Industry come into play in commercial and financial considerations, and it is necessary to build decisions that will guarantee to optimize the profit or cost or time etc. Since Simplex algorithm for solving the general optimization problem (1) have been developed, experienced, and effectively applied to a lot of vital problems in scientific and economic interest. However, in spite of the large number of optimization methods, there is no worldwide method for solving all optimization problems.

3.0 Decomposition strategies

When a problem becomes too large or too challenging to solve within reasonable time, decomposition approaches may be applied if the right problem structure is present. A problem is called *trivially parallelizable* if the subproblems can be solved completely independent of each other. Consider for example the following optimization problem

$$\text{Min } f(x_1,x_2,\ldots, x_n)=\text{Min } f_1(x_1)+f_2(x_2)+\ldots+f_n(x_n)$$

This problem is trivially parallelizable since the subproblems

$$\text{Minimize } f_i(x_i) \qquad \forall i$$

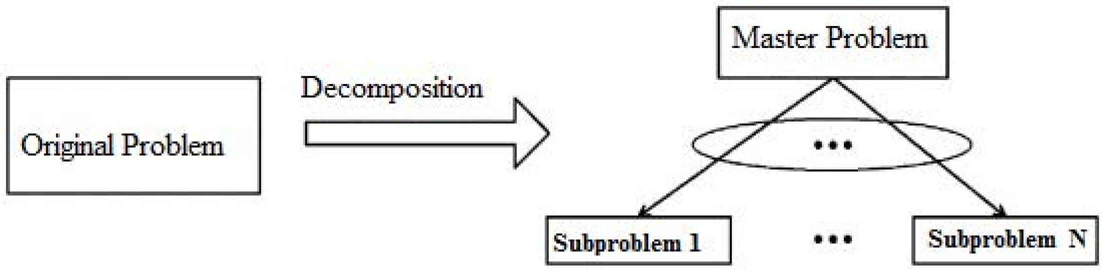

Figure 1: General Decomposition Structure

can be solved independently of each other. Inverting the block diagonal matrix (1) is another trivially parallelizable example. More interesting situations occur when there is a coupling between the subproblems, and that is the situation which decomposition methods are trying to solve.

3.1 Primal decomposition

Consider an unconstrained optimization problem that splits into two subproblems. In order to considering minimization problem, is of the form

$$\text{Minimize } f(x) = f_1(x_1, y) + f_2(x_2, y) \qquad (2)$$

where $x = (x_1, x_2, y)$ are the variable. Even if the dimension doesn't considerable here, it is used to think y having relatively small dimension and of x_1 and x_2 as having quite high dimension. The objective is more or less block separable with respect to x_1 and x_2; in fact, if the subvector y is predetermined, then the problem becomes separable with respect to x_1 and x_2, and the solution is obtained by solving the subproblems independently. This is the reason, y is called the complicating variable, since if y is fixed, the problem decomposes two or more subproblems. Thus study that the proposed problem becomes separable when y is fixed.

Denote $\varphi_1(y)$ and $\varphi_2(y)$ are the optimal solution of the following problems

Minimize $f_1(x_1, y)$ (3)
Minimize $f_2(x_2, y)$ (4)

Note that if the functions f_1 and f_2 are convex and so are φ_1 and φ_2. Treating (3) and (4) are the subproblem1 and subproblem 2. Thus the original problem (2) is equivalent to the problem

Minimize $\varphi_1(y) + \varphi_2(y)$ (5)

The variables of the main problem are the coupling or complicating variables of the original problem. The sum of the optimal values of the subproblems provides objective of the main problem. An iterative method used such as subgradient method solves the problem (2) such as the subgradient method. Each iterations are required to solve two subproblems parallel in order to evaluate $\varphi_1(y)$ and $\varphi_2(y)$ and their gradients or subgradients. However it is done in sequential order, there will be significant savings if the computational difficulty of the problems grows more than linearly with respect to the size of the problem. For those cases the associated subproblems are solved. Thus, there is a subgradient of f_1 which of the form $(0, g_1)$, and not unexpectedly, g_1 is a subgradient of φ_1 at y.

The same procedure can be adopted to find a subgradient $g_2 \in \partial\varphi_2(y)$ Then the subgradient of $\varphi_1 + \varphi_2$ at y is $g_1 + g_2$. The main problem can be solved by a variety of methods, together with bisection (if the dimension of y is one), gradient or quasi-Newton methods (if the functions are differentiable), or subgradient, cutting-plane, or ellipsoid methods (if the functions are nondifferentiable). This fundamental decomposition method is called primal decomposition as the foremost algorithm manipulates primal variables. When a subgradient method is used to solve the main problem, it reveals a extremely uncomplicated primal decomposition algorithm.

The same procedure can be adopted to solve the subproblems (possibly in parallel).

Find x_1' that optimizes $f_1(x_1, y)$, and a subgradient $g_1 \in \partial\varphi_1(y)$.
Find $x_{2'}$ that optimizes $f_2(x_2, y)$, and a subgradient $g_2 \in \partial\varphi_2(y)$.
Upgrade coupling variable. $y = y - \alpha_k(g_1 + g_2)$.

Here α_k is a step size. Now the interpretation of the above decomposition as follows.

The basic primal decomposition method can be described as the in a number of ways such. Add separable constraints, i.e., constraints of the form $x_1 \in C_1$, $x_2 \in C_2$. The possibility of this case that may obtain $\varphi_i(y) = \infty$ (i.e., $y \notin dom\ \varphi$) for some of y. For that case it finds a cutting-plane that separates y is from *dom φ*, to use in the main algorithm.

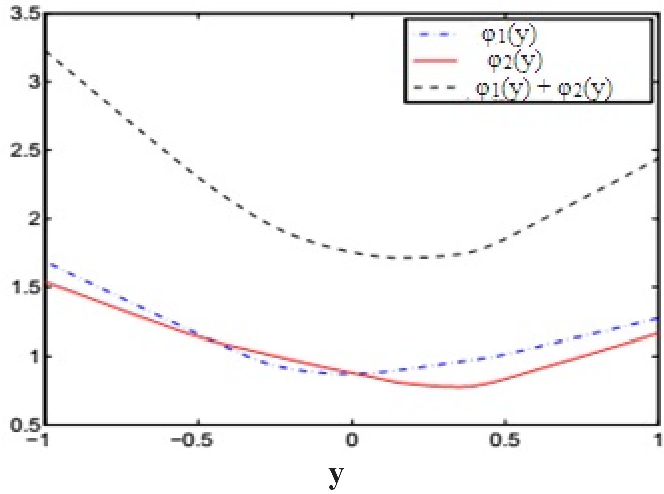

Figure 2: Objective function of the main problem and decomposed components as a function of y

4.0 Dual decomposition

Decomposition techniques are applied to the problem (2) after introducing a few variables y_1 and y_2, and the variables worked with the dual problem. Initially express the another problemequality constraint and a new variable as

$$\text{Minimize} \quad f(x) = f_1(x_1, y_1) + f_2(x_2, y_2)$$

subject to

$$y_1 = y_2, \qquad (6)$$

Here it introduced a limited description of the complicating variable y, along with a stability constraint that requires the two neighboring versions to be equal. Note that the objective is now separable, with the separation of variables (x_1, y_1) and (x_2, y_2).

Now the general form dual problem is given in the form of Lagrangian function. Thus the Lagrangian function is

$$L(x_1, y_1, x_2, y_2, \lambda) = f_1(x_1, y_1) + f_2(x_2, y_2) + \lambda^T y_1 - \lambda^T y_2,$$

which is separable. The dual function is

$$g(\lambda) = g_1(\lambda) + g_2(\lambda),$$

where

$$g_1(\nu) = \inf_{x_1, y_1}(f_1(x_1, y_1) + \nu^T y_1)$$

$$g_2(\nu) = \inf_{x_2, y_2} (f_2(x_2, y_2) + \nu^T y_2)$$

Note that g_1 and g_2 can be solved entirely in parallel those g_1 and g_2 can be expressed in conjugates of f_1 and f_2:

$$g_1(\lambda) = -f_1^*(0, -\lambda), \qquad g_2(\lambda) = -f_2^*(0, \lambda).$$

Then the dual problem can be written as

$$\text{Maximize } g_1(\lambda) + g_2(\lambda) = -f_1^*(0, -\lambda) - f_2^*(0, \lambda), \tag{7}$$

with variable λ. The equation (7) provides main problem in dual decomposition. The original algorithm evaluates the problem using some subgradient, cutting-plane, or other method. To find a subgradient of $-g_1$ (or $-g_2$) is easy. Find x_1' and y_1' that

$$\text{Minimize } f_1(x_1, y_1) + \lambda^T y_1$$

over x_1 and y_1. Then a subgradient of $-g_1$ at λ is $-y_1$. The same work can be adopted for x_2' and y_2'. Thus

$$\text{Minimize } f_2(x_2, y_2) - \lambda^T y_2$$

over x_2 and y_2, then a subgradient of $-g_2$ at λ is known by y_2'. Thus, a subgradient of the negative dual function −g is known by $y_2' - y_1'$, which is nothing more than the stability constraint left over. If the main problem is evaluated by using a subgradient manner, then the dual decomposition algorithm has a extremely simple form.

The same procedure can be adopted to solve the subproblems (possibly in parallel).

Find x_1 and y_1 that minimize $f_1(x_1, y_1) + \lambda^T y_1$. Find x_2 and y_2 that minimize $f_2(x_2, y_2) - \lambda^T y_2$.

Upgrade dual variables (prices).

$$\lambda = \lambda - \alpha_k(y_2 - y_1).$$

Here the step size α_k can be preferred in many ways. If the step size is constant, then the dual function g is differentiable, provided it is small sufficient. A further option in this case is to take out a line search on the dual objective. If the dual function is nondifferentiable, can use a diminishing nonsummable step size, such as $\alpha_k = \alpha/k$. At each step of the dual decomposition technique, provides a lower bound on p*, the optimal solution of the main problem, given by

$$p^* \geq g(\lambda) = f_1(x_1, y_1) + \lambda^T y_1 + f_2(x_2, y_2) - \lambda^T y_2. \tag{8}$$

where x_1, y_1, x_2, y_2 are the iterates. In general, the iterations are not feasible for the main problem, i.e., we have $y_2 - y_1 \neq 0$. A logical deduction of a feasible point can be constructed from this iterate as

$$(x_1, y') \qquad (x_2, y')$$

where $y' = (y_1 + y_2)/2$. In other words, replace y_1 and y_2 (which are different) with their average value. (The average is the projection of (y_1, y_2) onto the feasible set $y_1 = y_2$.) This gives an upper bound on p*, given by

$$p^* \leq f_1(x_1, y') + f_2(x_2, y'). \qquad \text{------ (9)}$$

A improved feasible point can be establish by replacing y_1 and y_2 with their average, and then evaluating the two subproblems (3) and (4) encountered in primal decomposition, i.e., by solving $\varphi_1(y') + \varphi_2(y')$. This gives the optimal bound

$$p^* \leq \varphi_1(y') + \varphi_2(y'). \qquad \text{------- (10)}$$

An interesting economic interpretation has dual decomposition. While imagine two distinct units, each consists of some cost function, but also some coupled variables and its own limited variables. Hence y_1 and y_2 as the amounts of some resources generated by the first unit and second unit respectively. Then, the stability condition $y_1 = y_2$ earns that supply and demand are equal (balanced).The amount of resources transferred from one unit to another is described in primal decomposition., and updates these set of transfer amounts until the total cost is optimized.

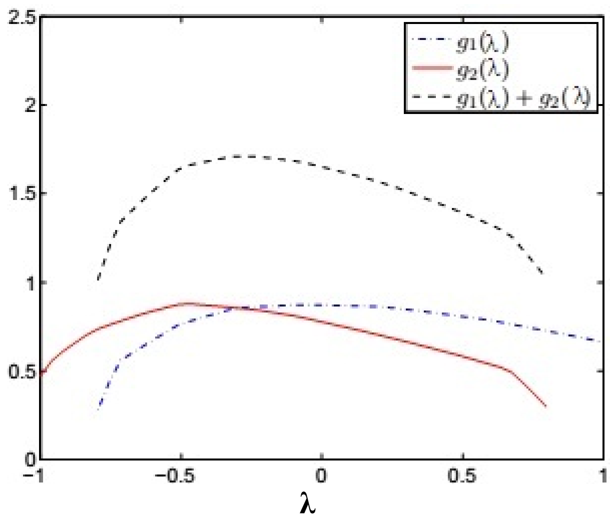

Figure 3: Dual Functions versus λ

Dual decomposition can deduce λ as a set of prices for the resources. The deal algorithm transfers the sets the prices, not the real amount from one unit to the other.

Then, every unit separately operates in such a way that its cost is minimized. The dual decomposition main algorithm adjusts the prices in order to carry the supply into consistency with the demand. In economics, the main dual algorithm is called a price adjustment algorithm.

There is one subtlety in dual decomposition. For prices λ^*, there are some accepted procedures for regularizing the subproblems that can work extremely strong in practice. In primal decomposition the subproblem can come across infinite values correspondingly dual decomposition can have $g_i(\lambda) = -\infty$. This can happen for some values of λ, if the functions f_i grow only linearly in y_i. In similar case produce a cutting-plane that separates the present price vector from *dom* g_i, and use this cutting-plane to renew the price vector.

Numerical Example 4.1

Consider the single coupling or complicating variable primal decomposition with a simple example. The problem is of the form

$$\text{Minimize } f(x) = f_1(x_1, y) + f_2(x_2, y)$$

Where f_1 and f_2 are piecewise-linear convex functions of (x_1, y) and (x_2, y). Figure 3 classifieds that g_1, g_2, and $g_1 + g_2$ as functions of λ. The optimal value of λ is obtained by $\lambda^* \approx -0.27$.

Figure 4 reveals that the growth of a bisection method for maximizing $g_1(\lambda)+g_2(\lambda)$, start at the opening interval $[-1, 1]$. At each step, the two subproblems are solved separately (in parallel), using the current price λ. This figure reveals that two upper bounds on p^*. The worse one is $f_1(x_1, y') + f_2(x_2, y')$; the smaller (better) one is $\varphi_1(y') + \varphi_2(y')$ (obtained by solving the subproblems).

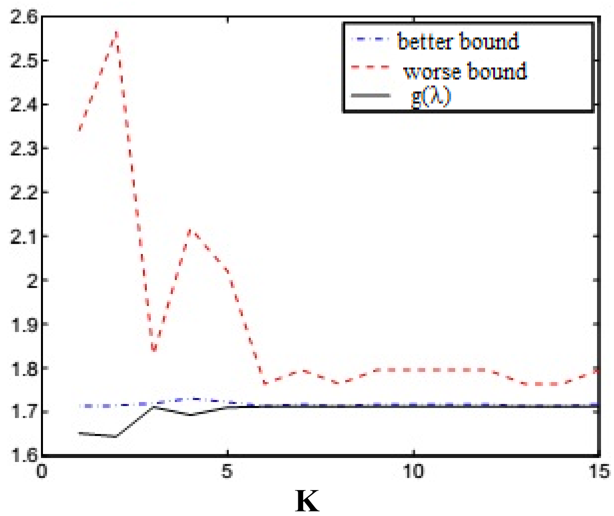

Figure 4: Convergence of dual function (lower bound), and simple and better upper bounds

4.1 The method of Lagrange Multipliers:
Lagrange multiplier introduces one or more variable λ, into the problem replacing the constraint into the objective function. This variable is known as Lagrange multiplier and has an significant economic understanding which is lies between zero and one. Lagrangian is the method of Lagrange relies on maximizing a coupled function. The Lagrangian function can be formed by adding λ times the constraint to the objective function and maximizing over the control variables and also the Lagrangian Multiplier.

Suppose the given problem is of the form
Max f(x , y)
Subject to
$g(x,y) \leq m$

Then the Lagrangian function is of the form,
$L[x, \lambda] = \text{Max } f(x, y) + \lambda (m - g(x, y))$
where x is the Lagrangian variable,
λ be the Lagrangian multiplier
g(x , y) be the constraint of the problem
m be the constant

In order to maximize the Lagrangian function, partial derivatives are taken with respect to the control variables and set them is equal to zero and solve them simultaneously:
$[x] : f_x - \lambda g_x$
$[y] : f_y - \lambda g_y$
$[\lambda] : m - g(x, y)$

And to obtain the optimal values of the control variables x*, y*, λ*. Note that here constraints are treated as explicitly. Thus there is no chain rule effect in the first order conditions. In this process first order conditions is that treat all variables except the control variable at hand as constant. That is the reason for the partial derivatives used and not total derivatives. The solution satisfies the constraint because in this case constraints are also one of the conditions so that must be explicitly satisfied.

4.2. Lagrangian Relaxation
Lagrangian relaxation (LR) is described in Beasley (1993). The LR idea is behind to attach Lagrangian multipliers to some of the constraints, relax these into the objective function, and then solve the original problem. When applying LR technique, the problem can in some cases be divided into smaller isolated subproblems, either naturally, i.e. by relaxing "common constraints" or by splitting variables and relaxing their equality binding, provided the objective and the relaxed constraints are additive in these variables. Different approaches can be used to find the values of the Lagrangian multipliers, for instance a sub-gradient method. Lagrangian Relaxation is a procedure to obtain good upper bounds for a problem (maximization problem), as

lower bounds usually can be found using a heuristic. The starting point when using the methods is a modified version of the original problem one is trying to solve, which is known as the Lagrangian Relaxation. The main idea is to eliminate complicating constraints from the original problem, but to punish the objective function for breaking the constraints that have been removed. This is done by moving the relaxed constraints into the objective function and then multiplying them with a Lagrange multiplier (Fisher, 1981). How this is done for a typical optimization problem is illustrated in equation 2.

Max cx	$\max\ cx + \lambda(h - Hx)$
subject to	*subject to,*
$Ax = b$	$Ax = b$
$Hx \le h$	$x \ge 0$
$x \ge 0$	$\lambda \ge 0$ ---------- (11)

Given that $\lambda > 0$ and that a constraint has been removed going from the formulation on the left to the formulation on the right in 1, can be derived by the inspection of Lagrangian Relaxation will always be an upper bound on the original problem. By adjusting the Lagrange multiplier through a series of iterations, the hope is that a better upper bound can be found by finding the "correct price" of breaking the relaxed constraint(s). The central decisions in an implementation of Lagrange Relaxation are which constraint(s) to relax and how to update the Lagrange multipliers. The aim should be to obtain a Lagrangian Relaxation that easily can be solved. Two general techniques are available for finding the Lagrange multipliers – subgradient optimization and multiplier adjustment (Beasley, 1993). Subgradient optimization method is an iterative procedure that attempts solve the dual to the Lagrangian relaxation. That is, it tries to maximize the lower bound value obtained from the Lagrangian Relaxation. Multiplier adjustment is a heuristic approach that generates a starting set of Lagrange Multipliers, tries to improve them in some systematic way and then repeats the procedure if improvements are made.

Numerical Example 4.2.1

Minimize $(x-2)^2 + 2(y-1)^2$

Subject to $x + 4y \le 3$

$x \ge y$

The corresponding standard form is

Maximize $-(x-2)^2 - 2(y-1)^2$

Subject to $x + 4y \le 3$

$-x + y \le 0$

$L(x, y, \lambda_1, \lambda_2) = -(x-2)^2 - 2(y-1)^2 + \lambda_1(3 - x - 4y) + \lambda_2(0 + x - y)$

Which gives the optimality condition

$-2(x - 2) - \lambda_1 + \lambda_2 = 0$

$-4(y - 1) - 4\lambda_1 - \lambda_2 = 0$

$\lambda_1 (3 - x - 4y) = 0$

$\lambda_2 (x - y) = 0$

$x + 4y \leq 3$
$-x + y \leq 0$
$\lambda_1 , \lambda_2 \geq 0$

Since here are two complementary conditions, there are four cases to check:

i. $\lambda_1 = 0$, $\lambda_2 = 0$; provides x = 2, y = 1 which is not feasible.
ii. $\lambda_1 = 0$, $x - y = 0$; provides x= 4/3, y = 4/3, λ_2 = -4/3 which is not feasible.
iii. $\lambda_2 = 0$, $3 - x - 4y = 0$ provides x = 5/3, y = 1/3, λ_1 = 2/3 which is feasible.
iv. $3–x–4y = 0$, $x–y = 0$ provides x = 3/5, y = 3/5, λ_1 =22/25, and λ_2 = -48/25 hence not feasible.

Since from the above cases it obtained that the optimal solution, x = 5/3, y = 1/3.

5.0 Conclusion:

An analysis has been made to show that the Lagrangian Decomposition model gives the appropriate solution. Lagrangian functions give the optimum solution for the proposed decomposition techniques with respect to penalty factors. An algorithmic approach is designed for the Lagrangian Relaxation method. Lagrangian dual gives the optimal solution for the subproblems which optimizes the main problem. The aim of the paper is to identify the control and uncontrolled parameters of the system which are framed as equality, inequality constraints. Particularly the Lagrangian multipliers added in the objective function of the Lagrangian problem which is acting as "penalty factors", based on the parameters of the system. The objective function of the model includes the relaxed constraint gives the optimal solution. On decomposition technique the Lagrangian Multipliers (λ) are used to relax one or more constraints in such a way that main problem gets optimal solution. The proposed model gives the optimal solutions for the optimization problem.

Reference:

1. Bissan Ghaddar,"*A Lagrangian decomposition approach for the pump scheduling problem in water networks*", European Journal of Operational Research 241(2015), pp 490-501.
2. Leslie Chandrakantha (2012), A modern approach "*using Excel Solver for Optimization Problem*"
3. Daniel E. Marthaler(2013), *"An Overview of Mathematical Methods for Numerical optimization"* , Springer Science+Business Media Dordrecht, pp 31-53
4. M. Chiang, S. H. Low, A. R. Calderbank, and J. C. Doyle, *"Layering as optimization decomposition," Proc. IEEE*, Dec. 2006, to be published.
5. Hakan Terelius , *"Distributed Multi-Agent Optimization via Dual Decomposition"* , Main's thesis, Stockholm , Sweden 2010

6. Haldun Aytug, "*Feature selection for support vector machines using Generalized Benders Decomposition*", European Journal of Operational Research 244(2015), pp 210-218
7. J. N. Hooker (2007)," *Planning and scheduling by logic-based Benders decomposition",* Operations Research, 55(3), pp 588 – 602.
8. Geoffrion A. M., "*Lagrangean Relaxation for Integer Programming*,"Mathematical Programming Study*,* *2*, 82 – 114 (1974).
9. M.L. Fisher, *"The Lagrangian relaxation method for solving integer programming problems".* Management Science, 27:1–18, 1981.
10. Alexander M and Michel Collins,"*A tutorial on dual decomposition and Lagrangian Relaxation for Inference in Natural Language Processing*" an online source (2011).
11. Matthew Galati," Decomposition methods for Integer Linear Programming", a doctoral thesis of Lehigh University, January 2010.
12. G. B. Dantzig and P. Wolfe. Decomposition principle for linear programs. Operations Research, 8:101–111, 1960.
13. D. P. Bertsekas. Nonlinear Programming. Athena Scientific, second edition, 1999.
14. Vidar Gunnerud *'On decomposition and piecewise linearization in petroleum production optimization'* , Doctoral thesis at Norwegian University of Science and Technology, 2011 : 164