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Pseudo KU-algebras and their applications in topology

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Abstract

In this paper, we introduce pseudo KU-algebras and we also apply topological space for the pseudo KU-algebra. Furthermore, some fundamental properties of the topological pseudo KU-algebras are investigated.

AMS subject classification:
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1. Introduction

There has been some interest in studying topologies of classes of algebras recently. The topological notions of algebras (BCC/BCK/BCI-algebras) have been given in [1, 2, 3]. In 1998, D.S. Lee and D.N. Ryu introduced the notion of topological BCK-algebra and gave some topological properties of this structure. They gave a filter base generating a BCK-algebra topology which is a fundamental neighbourhood system of zero for that topology. With the same concept, the notion of topological BCI-algebras was introduced by Y.B. Jun, X.L. Xin and D.S. Lee in 1999. They gave a characterization of a topological

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BCI-algebra in terms of neighbourhoods and showed that a topological BCI-algebra is Hausdorf if and only if $\{0\}$ is closed in the topological BCI-algebra, and also gave a filter base generating a BCI-topology. Later in 2008, the notion of topologies of BCC-algebras was introduced by S.S. Ahn and S.H. Kwon.

An algebra $X = (X, \cdot, 0)$ of type (2,0) is a KU-algebra if the following identities are satisfied:

(i) $(x \cdot y) \cdot [(y \cdot z) \cdot (x \cdot z)] = 0$,

(ii) $0 \cdot x = x$,

(iii) $x \cdot 0 = 0$ and (iv) $x \cdot y = 0 = y \cdot x$ implies $x = y$, for all $x, y, z \in X$.

KU-algebras were originally introduced by C. Prabpayak and U. Leerawat in 2009 (see [5]). In this paper, we introduce another class of algebra called a pseudo KU-algebra (PKU-algebra) as an extension of KU-algebras, and also discuss the notions of ideals, congruences and quotient algebras. Finally, we give a topology on PKU-algebras and investigate some topological properties on PKU-algebras.

2. Materials and Methods

From now on, let X be a non-empty set.

Definition 2.1. By a pseudo KU-algebra (briefly, PKU-algebra) X we mean an algebra $(X, \cdot, 0)$ of type $(2, 0)$ with a binary operation $\cdot$ satisfying the following identities:

(P1) $(x \cdot y) \cdot [(y \cdot z) \cdot (x \cdot z)] = 0$,

(P2) $0 \cdot x = x$,

(P3) $x \cdot y = 0 = y \cdot x$ implies $x = y$,

for all $x, y, z \in X$.

We often write X instead of $(X, \cdot, 0)$ for a PKU-algebra and write xy instead of $x \cdot y$ in brevity.

Example 2.2. Let $X = \{0, 1, 2\}$ be a set with a binary operation $\cdot$ defined as follows:

$\cdot$	0	1	2
0	0	1	2
1	1	0	1
2	0	1	0

One can easily check that the set $(X, \cdot, 0)$ is a PKU-algebra.

Let X be a PKU-algebra. Then we have the following properties: for all $x, y, z \in X$,

(i) $xx = 0$,

(ii) $x[(xy)y] = 0$,

(iii) $xy = yz = 0$ implies $xz = 0$,

(iv) if $(xy)y = 0$, then X is a KU-algebra.

For elements x and y in a PKU-algebra X, we define a relation $\leq$ on X as follows; $y \leq x$ if and only if $xy = 0$. Then $(X, \leq)$ is a partially ordered set.

Proposition 2.3. For any PKU-algebra X we have the following properties: for all $x, y, z \in X$,

(i) $x \leq y$ implies $yz \leq xz$,

(ii) $x \leq y$ implies $zx \leq zy$.

Proof. Let $x \leq y$. This implies $yx = 0$. From (P1) we obtain $(yx)[(xz)(yz)] = 0$ and $(zy)[(yx)(zx)] = 0$. Since $yx = 0$, we get that $(xz)(yz) = 0$ and $(zy)(zx) = 0$ by (P2), which mean $yz \leq xz$ and $zx \leq zy$. ■

Proposition 2.4. For any elements x, y, z in PKU-algebra X, we have that $x(yz) = y(xz)$.

Proof. The proof is similar to [4, Lemma 2.6]. ■

Definition 2.5. Let X be a PKU-algebra. Then

(i) A non-empty subset A of X is called PKU-subalgebra of X if for all $x, y \in A$, $xy \in A$.

(ii) A non-empty subset A of X is called a PKU-ideal of X if it satisfies the following:

 (a) $0 \in A$,
 (b) for all $x, y, z \in X$, $x(yz) \in A$ and $y \in A$ implies $xz \in A$.

Note that any finite intersection of PKU-ideals of PKU-algebra is again a PKU-ideal.

Example 2.6. From the previous example, let $A = \{0, 1\}$ and $B = \{0, 2\}$. Then $(A, \cdot, 0)$ and $(B, \cdot, 0)$ are PKU-subalgebras of X. One can easily check that B is a PKU-ideal of X but A is not because $0(12) = 1 \in A$ and $02 = 2 \notin A$.

Remark 2.7.

(i) Let X be a PKU-algebra. We define the set $G(X) = \{x \in X \mid x(x0) = 0\}$ to be the G-part of X. Note that $G(X) \neq \phi$. For any $x, y \in G(X)$. We have that $x(x0) = 0$

and $y(y0) = 0$, so $y0 \leq y$. Using (ii) of Proposition 2.3 and Proposition 2.4 we obtain

$$\begin{aligned} x(y0) &\leq xy, \\ y(x0) &\leq xy, \\ x[y(x0)] &\leq x(xy), \\ y[x(x0)] &\leq x(xy), \\ y0 &\leq x(xy), \\ (xy)(y0) &\leq (xy)[x(xy)], \\ (xy)(y0) &\leq x[(xy)(xy)], \\ (xy)(y0) &\leq x0, \\ (y0)[(xy)(y0)] &\leq (y0)(x0), \\ (xy)[(y0)(y0)] &\leq (y0)(x0), \\ (xy)0 &\leq (y0)(x0). \end{aligned}$$

Since $(xy)[(y0)(x0)] = 0$, $(y0)(x0) \leq xy$. Then $(xy)0 \leq xy$. Therefore $(xy)[(xy)0] = 0$. Hence $xy \in G(X)$, and thus $G(X)$ is a PKU-subalgebra of X.

(ii) If S is a PKU-subalgebra of X then $G(X) \cap S = G(S)$.

Proposition 2.8. Let A be a PKU-ideal of a PKU-algebra X. Then for all $x, y \in X$, $xy \in A$ and $x \in A$ implies $y \in A$.

Proof. Taking $x = 0$ in (b) of Definition 2.5. ■

Let A be an ideal of a PKU-algebra X. A binary relation $\sim_A$ on X is defined by $x \sim_A y$ if and only if $xy \in A$ and $yx \in A$.

Theorem 2.9. Let A be a PKU-ideal of a PKU-algebra X. Then $\sim_A$ is a congruence relation on X.

Proof. $\sim_A$ is clearly reflexive and symmetric. Now let $x \sim_A y$ and $y \sim_A z$. Then we have $xy, yx, yz, zy \in A$. By (P1) we obtain $(xy)[(yz)(xz)] = 0 \in A$ and $(zy)[(yx)(zx)] = 0 \in A$. Hence $xz, zx \in A$ by Proposition 2.8 which implies $x \sim_A z$. Therefore $\sim_A$ is transitive, and thus $\sim_A$ is an equivalent relation.

Suppose $a \sim_A b$ and $x \sim_A y$. Then $ab, ba, xy, yx \in A$. By (P1) we have $(ab)[(bx)(ax)] = 0 \in A$ and $(ba)[(ax)(bx)] = 0 \in A$. Using Proposition 2.8, this implies $(bx)(ax) \in A$ and $(ax)(bx) \in A$, so $ax \sim_A bx$. We also have $(bx)[(xy)(by)] = 0 \in A$ and $(by)[(yx)(bx)] = 0 \in A$. Since A is an ideal, we obtain $(bx)(by) \in A$ and $(by)(bx) \in A$. Therefore $bx \sim_A by$. Since $\sim_A$ is transitive, $ax \sim_A by$. Hence $\sim_A$ is a congruence relation on X. ■

The equivalence class of an element $a \in X$ is denoted as the set

$$[a]_A = \{x \in X \mid a \sim_A x\}.$$

Proposition 2.10. Let A be a PKU-ideal of a PKU-algebra X. Then we have the following:

(i) $a \in [a]_A$,

(ii) $a \sim_A b$ if and only if $[a]_A = [b]_A$,

(iii) every two equivalence classes $[a]_A$ and $[b]_A$ are either equal or disjoint.

Proof.

(i) is obvious.

(ii) For the first direction we let $a \sim_A b$. Then we have $ab \in A$ and $ba \in A$. Let $x \in [a]_A$. We then obtain $xa \in A$ and $ax \in A$. So $(xa)[(ab)(xb)] = 0 \in A$ and $(ba)[(ax)(bx)] = 0 \in A$. Thus $xb \in A$ and $bx \in A$ by Proposition 2.8. Hence $x \in [b]_A$. Therefore $[a]_A \subset [b]_A$. Similarly, we obtain $[b]_A \subset [a]_A$. Thus $[a]_A$ and $[b]_A$ are equal.

On the other hand, we suppose that $[a]_A = [b]_A$. By (i), $a \in [a]_A = [b]_A$. We immediately get that $ab \in A$ and $ba \in A$ which implies $a \sim_A b$.

(iii) Suppose that $[a]_A \neq [b]_A$ and $[a]_A \cap [b]_A$ is not empty. Then there exists an element $x \in X$ such that $x \in [a]_A \cap [b]_A$. This means x is contained in both $[a]_A$ and $[b]_A$, so we have $a \sim_A x$ and $x \sim_A b$. Since $\sim_A$ is transitive, $a \sim_A b$. By (ii), they are equal, which is a contradiction. ■

Let A be a PKU-ideal of a PKU-algebra $(X, \cdot, 0)$. We define the quotient set

$$X/A = \{[a]_A \mid a \in X\}$$

and define a binary operation $*$ on X/A by $[a]_A * [b]_A = [a \cdot b]_A$.

Theorem 2.11. Let A be a PKU-ideal of a PKU-algebra X. Then $(X/A, *, [0]_A)$ is a PKU-algebra. (for brevity, call X/A a PKU-quotient algebra).

Proof. Firstly we show that $*$ is well-defined on X/A. Suppose that $[a]_A = [b]_A$ and $[x]_A = [y]_A$. Then $a \sim_A b$ and $x \sim_A y$. Since $\sim_A$ is a congruence relation, $ax \sim_A by$. Hence $[a]_A * [x]_A = [ax]_A = [by]_A = [b]_A * [y]_A$, and therefore $*$ is well-defined.

Now let $x, y, z \in X$. Then $([x]_A * [y]_A) * [([y]_A * [z]_A) * ([x]_A * [z]_A)] = [(xy)[(yz)(xz)]]_A = [0]_A$ and $[0]_A * [x]_A = [0x]_A = [x]_A$. If $[x]_A * [y]_A = [0]_A = [y]_A * [x]_A$, then we have that $xy \in [xy]_A = [x]_A * [y]_A = [0]_A$, so $xy \in A$. Similarly, we obtain $yx \in A$. This means $x \sim_A y$. Therefore $[x]_A = [y]_A$ by Proposition 2.10 (ii). Hence X/A is a PKU-algebra. ■

3. Results and Discussion

Let X be a set. A topology on X is a collection $\Gamma \subset P(X)$ of subsets of X which satisfies the following:

(i) Γ contains ϕ and X,

(ii) Γ is closed under arbitrary unions, i.e. if $U_i \in \Gamma$ for $i \in I$ then $\bigcup_{i \in T} U_i \in \Gamma$,

(iii) Γ is closed under finite intersections, i.e. if $U_1, U_2 \in \Gamma$ then $U_1 \cap U_2 \in \Gamma$.

A topological space (X, Γ) is a set X together with a topology Γ on it. The elements of Γ are called open subsets of X. A subset $F \subset X$ is said to be closed if its complement F^c is open. A subset N containing a point $x \in X$ is called a neighbourhood of x if there exists an open set U with $x \in U \subset N$. Hence an open neighbourhood of x is an open subset containing x.

Let X be a PKU-algebras. Let $\mathcal{A}$ be an arbitrary collection of PKU-ideals of X which is closed under intersections. Then we define

$$\Gamma_{\mathcal{A}} = \{G \subset X \mid \forall x \in G, \exists A \in \mathcal{A}, [x]_A \subset G\}. \tag{3.1}$$

If $\mathcal{A} = \{A\}$, we denote it by Γ_A.

For any $x \in X$, one can easily see that $[x]_A$ in contained in $\Gamma_{\mathcal{A}}$ and x always lies in $[x]_A$. Hence $\Gamma_{\mathcal{A}}$ is not empty.

Throughout this section, $\Gamma_{\mathcal{A}}$ refers to the set which is defined by (3.1).

Theorem 3.1. Let X be a PKU-algebras. Then $\Gamma_{\mathcal{A}}$ is a topology on X.

Proof. It is clear that ϕ and X are contained in $\Gamma_{\mathcal{A}}$ and also clear that $\Gamma_{\mathcal{A}}$ is closed under arbitrary unions. To show that $\Gamma_{\mathcal{A}}$ is closed under finite intersection, we let $G, H \in \Gamma_{\mathcal{A}}$, and let $x \in G \cap H$. Then there exist PKU-ideals A and B in $\mathcal{A}$ such that $[x]_A \subset G$ and $[x]_B \subset H$. We then obtain $[x]_{A\cap B} \subset [x]_A \cap [x]_B \subset G \cap H$. Since $A \cap B$ is a PKU-ideal of X, $G \cap H \in \Gamma_{\mathcal{A}}$. Therefore $\Gamma_{\mathcal{A}}$ is a topology on X. ■

If $(X, \Gamma_{\mathcal{A}})$ is a topological space and X is a PKU-algebra, then X is said to be a topological PKU-algebra.

Proposition 3.2. Let A be a PKU-ideal of a PKU-algebra. Then

(i) $A = \bigcup\{[a]_A \mid a \in A\}$,

(ii) $A^c = \bigcup\{[a]_A \mid a \in A^c\}$.

Proof. (i) is obvious. (ii) Let $x \in A^c$. Then we have $x \in [x]_A \subset \bigcup\{[a]_A \mid a \in A^c\}$. Thus $A^c \subset \bigcup\{[a]_A \mid a \in A^c\}$. To show that $A^c \supset \bigcup\{[a]_A \mid a \in A^c\}$, we let $y \in [a]_A$

for some $a \in A^c$. This implies $y \sim_A a$, and we obtain $ya \in A$. If $y \in A$, then $a \in A$ by Proposition 2.8, a contradiction. Therefore $y \in A^c$. Thus $A^c \subset \bigcup\{[a]_A \mid a \in A^c\}$. So (ii) holds. ■

In topology, a clopen set in a topological space is a set which is both open and closed. A topological space X is connected if and only if X itself and empty set are only clopen subsets of X.

Theorem 3.3. Let X be a topological PKU-algebra. Then

(i) any PKU-ideal in $\mathcal{A}$ is a clopen subset of X.

(ii) for any $x \in X$ and $A \in \mathcal{A}$, $[x]_A$ is a clopen subset of X.

Proof.

(i) Let $A \in \mathcal{A}$. Obviously $A \in \Gamma_{\mathcal{A}}$. Thus A is open. We then show that A^c is open. Let $x \in A^c$. Follows from Proposition 3.2 (ii) we obtain that $[x]_A \subset A^c$. Then $A^c \in \Gamma_{\mathcal{A}}$. Hence $[x]_A$ is clopen.

(ii) It is clear that $[x]_A$ is open. It suffices to prove that $([x]_A)^c$ is open. Let $y \in ([x]_A)^c$. Then $xy \in A^c$ or $yx \in A^c$. Without loss of generality, we assume that $xy \in A^c$. It remains to show that $[y]_A \subset ([x]_A)^c$. If $z \in [y]_A$, then $z \sim_A y$. Since $\sim_A$ is a congruence relation, $xz \sim_A xy$. By Proposition 3.2 (ii) we know that $A^c = \bigcup\{[a]_A \mid a \in A^c\}$. Thus there exists some $a \in A^c$ such that $xy \in [a]_A$. It follows that $xy \sim_A a$. Hence $xz \sim_A a$ which means $xz \in [a]_A \subset A^c$. Therefore $z \in ([x]_A)^c$. So $[y]_A \subset ([x]_A)^c$. ■

Corollary 3.4. Let X be a PKU-algebra. Then $(X, \Gamma_{\mathcal{A}})$ is not a connected space.

Theorem 3.5. Let X be a topological PKU-algebra. Then we have the following:

(i) if $B = \bigcap\{A \mid A \in \mathcal{A}\}$, then $\Gamma_{\mathcal{A}} = \Gamma_B$,

(ii) for any PKU-ideals A and B of X, if $A \subset B$, then B is clopen in the topological PKU-algebra (X, Γ_A),

(iii) for any PKU-ideals A and B of X, if $A \subset B$, then $\Gamma_B \subset \Gamma_A$.

Proof.

(i) Let $G \in \Gamma_{\mathcal{A}}$, and let $x \in G$. Then there exists a PKU-ideal A of X in $\mathcal{A}$ such that $[x]_A \subset G$, and $B \subset A$. So we obtain $[x]_B \subset [x]_A \subset G$. This implies that $G \in \Gamma_B$. On the other hand, let $H \in \Gamma_B$, and let $y \in H$. Then $[y]_B \subset H$. Since $\mathcal{A}$ is closed under intersection (i.e. $B \in \mathcal{A}$), we immediately get that $H \in \Gamma_{\mathcal{A}}$. Hence (i) holds.

(ii) Follows from (i) by considering $\mathcal{A} = \{A, B\}$.

(iii) Let $A \subset B$, and let us denote $\mathcal{A} = \{A\}$ and $\mathcal{B} = \{B\}$. Assume that $G \in \Gamma_{\mathcal{B}}$ and $x \in G$. Then $[x]_B \subset G$. Since we know that $[x]_A \subset [x]_B$, so $G \in \Gamma_{\mathcal{A}}$. ■

Definition 3.6. Let $(X, \Gamma_{\mathcal{A}})$ be a topological PKU-algebra.

(i) Let Y be a subset of X. An open cover for Y is a collection $\mathcal{O}$ of open subsets of X such that $Y \subset \bigcup\{O : O \in \mathcal{O}\}$.

(ii) A subset Y of X is said to be compact if any open cover of Y has its finite subcover (i.e. for each open cover $\mathcal{O}$ for Y, there exist $\mathcal{O}_1, \ldots, \mathcal{O}_n \in \mathcal{O}$ such that $Y \subset \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n$).

(iii) $(X, \Gamma_{\mathcal{A}})$ is said to be totally bounded if for each $A \in \mathcal{A}$, there exist $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} [x_i]_A$.

Theorem 3.7. Let A be a PKU-ideal of a PKU-algebra X. Then the following are equivalent:

(i) the topological PKU-algebra (X, Γ_A) is compact,

(ii) the topological PKU-algebra (X, Γ_A) is totally bounded,

(iii) there exists $I = \{x_1, \ldots, x_n\} \subset X$ such that for all $a \in X$ there exist $x_i \in I$ $(i = 1, \ldots, n)$ with $ax_i \in A$ and $x_i a \in A$.

Proof. (i) ⇒ (ii) Denote $\mathcal{O} := \bigcup\{[x]_A \mid x \in X\}$. Then $X = \mathcal{O}$. By our assumption, there exist $[x_1]_A, \ldots, [x_n]_A$ in $\mathcal{O}$ such that $X \subset \bigcup_{i=1}^{n} [x_i]_A$. But we have $X = \mathcal{O}$, then $X = \bigcup_{i=1}^{n} [x_i]_A$.

(ii) ⇒ (iii) Since (X, Γ_A) is totally bounded, there exist $x_1, \ldots, x_n \in X$ such that $X = \bigcup_{i=1}^{n} [x_i]_A$. Now let $a \in X$. Then there exists x_i such that $a \in [x_i]_A$, which implies that $ax_i \in A$ and $x_i a \in A$.

(iii) ⇒ (i) Let $a \in X$. By our assumption, there exists $x_i \in I$ with $ax_i \in A$ and $x_i a \in A$. Hence $a \in [x_i]_A$, and thus $X = \bigcup_{i=1}^{n} [x_i]_A$. Now let $X = \bigcup O_\alpha$ where O_α are open sets of X. Then for any $x_i \in I$. There exists α_i such that $x_i \in O_{\alpha_i}$. Since $O_{\alpha_i} \in \Gamma_A$, $[x_i]_A \subset O_i$. Therefore $X = \bigcup_{i=1}^{n}[x_i]_A \subset \bigcup_{i=1}^{n} O_i$. So (X, Γ_A) is compact. ■

Theorem 3.8. Let A be a PKU-ideal of a PKU-algebra X. Then for any $x \in X$, $[x]_A$ is compact in a topological space (X, Γ_A).

Proof. Let $x \in X$ and let $\mathcal{O}$ be an open cover for $[x]_A$. Then $[x]_A \subset \bigcup\{O \mid O \in \mathcal{O}\}$. Since $x \in [x]_A$, so x is contained in O for some $O \in \mathcal{O}$. This means that $[x]_A \subset O$ since $\mathcal{O}$ lies in Γ_A. Therefore $[x]_A$ is compact. ■

4. Conclusion

We have introduced a new class of algebra called a pseudo KU-algebra. The notion of ideals of the pseudo KU-algebra has been introduced which can be used to construct a pseudo KU-quotient algebra. Then we have described the topological properties of the pseudo KU-algebra that a class of a pseudo KU-quotient algebra generated by element x in the pseudo KU-algebra respect to some ideal is actually an open neighbourhood of x. Finally we have given a topological space on the pseudo KU-algebra, called a topological pseudo KU-algebra, and investigated its related properties.

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