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A Class of Differential System with at most Two Algebraic Limit Cycles

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Abstract

A class of differential system of degree greater than or equal five is introduced, we show that under suitable assumptions, one or two algebraic limit cycles can occur, these limit cycles are analytically given.

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1. Introduction

The second part of sixteenth problem of Hilbert still persists a research area. It offers to find the maximum number of limit cycles of the differential system:

$$\begin{cases} \dot{x} = \dfrac{dx}{dt} = P(x, y) \\ \dot{y} = \dfrac{dy}{dt} = Q(x, y) \end{cases} \tag{1.1}$$

where P and Q are polynomials.

The analysis of the existence, number and stability of limit cycles, have been several articles (see for instance T.R. Blows and N.G. Lloyd [4], F. Dumortier, J. Llibre and J. Artés [7], H. Giacomini, J. Llibre and M. Viano [8], J. Chavarriga, H. Giacomini and J. Giné [5], X. Zhang [13]).

Generally, the exact analytical expressions of limit cycles for a given differential system are unknown, except in specific cases.

This paper is a contribution in that direction, determine the number of limit cycles and give their explicit form.

Motivated by some publications (C. Christopher [6], M. Abdelkadder [1], A. Bendjeddou and R. Cheurfa [2], [3], J. Llibre and Y. Zhao [10], J.Giné and M. Grau [9], J. Llibre, R. Ramírez, N. Sadovskaia [11], [12]) we will study the existence and the number of limit cycles of a class of planar differential system of degree greater than or equal five, and give their explicit form.

2. Introductory concepts

Let us recall some useful notions.

For $U \in \mathbb{R}[x, y]$, the algebraic curve $U = 0$ is called an invariant curve of the polynomial system (1.1), if for some polynomial $K \in \mathbb{R}[x, y]$ called the cofactor of the algebraic curve, we have

$$P(x, y)\frac{\partial U}{\partial x} + Q(x, y)\frac{\partial U}{\partial y} = KU \tag{2.1}$$

Simple analysis of equation (2.1) shows that when $\max(\deg P, \deg Q) = n$, the degree of the cofactor is at most $n - 1$ and that the curve $U = 0$ is formed by trajectories of the system (1.1). Also, if the curve $U = 0$ is nonsingular, the equilibrium points of the system that satisfy

$$\begin{cases} P(x, y) = 0 \\ Q(x, y) = 0 \end{cases} \tag{2.2}$$

are contained either in its unbounded components or are located on the curve $K = 0$.

A limit cycle $\gamma = \{(x(t), y(t)), t \in [0, T]\}$, is a $T-$periodic solution isolated with respect to all other possible periodic solutions of the system.

We construct here a multi-parameter planar differential system of degree greater than or equal five admitting the closed components of the curve

$$\Gamma = \left\{(x, y) \in \mathbb{R}^2 : ax^n + bx^{n-4} + cy^2 + dy + h = 0\right\}, n \geq 4 \tag{2.3}$$

as limit cycles if some conditions on the parameter are satisfied.

3. The main result

We consider a differential system

$$\begin{cases} \dot{x} = V(x, y)U(x, y) - U_y(x, y) \\ \\ \dot{y} = W(x, y)U(x, y) + U_x(x, y) \end{cases} \tag{3.1}$$

where $V(x, y)$ and $W(x, y)$ are polynomials of any degree, $U(x, y) = ax^n + bx^{n-4} + cy^2 + dy + h$ with a, c positive real and b negative real.

As a main result, we have the following theorem

Theorem 3.1. The system (3.1), when a, c are positive real, b is negative real, and $\frac{\partial V}{\partial x} + \frac{\partial W}{\partial y} > 0$ admits exactly: One limit cycle if $h < \frac{d^2}{4c}$, when n is even number greater than or equal four.

Two limit cycles if $\frac{d^2}{4c} < h < \frac{d^2}{4c} + \alpha_n$, when n is even number greater than four. One limit cycle if $h \in \left]\frac{d^2}{4c} - \alpha_n, \frac{d^2}{4c}\right[\cup \left]\frac{d^2}{4c}, \frac{d^2}{4c} + \alpha_n\right[$, when n is odd number greater than five.

One limit cycle if $\frac{d^2}{4c} - \alpha_n < h < \frac{d^2}{4c} + \alpha_n$ when $n = 5$.

$$\alpha_n = 4\left(\frac{n-4}{a}\right)^{\frac{n}{4}-1}\left(\frac{-b}{n}\right)^{\frac{n}{4}}$$

The limit cycles are represented by the curve Γ.

Proof. Simply mount the curve Γ is nonsingular formed by oval and it is an invariant of system (3.1) and the integral $\int_0^T K(x, y)dt$ is nonzero.

i) Γ is nonsingular of system (3.1) formed by oval.

We firstly prove that there is no singular point on the curve Γ. We recall that the curve Γ is nonsingular if the following system has no real solution

$$.\begin{cases} ax^n + bx^{n-4} + cy^2 + dy + h = 0 \\ 2cy + d = 0 \\ anx^{n-1} + b(n-4)x^{n-5} = 0 \end{cases} \tag{3.2}$$

The possible critical points on Γ are:

$$A_1\left(0, \frac{-d}{2c}\right), A_2\left(\sqrt[4]{\frac{-b(n-4)}{an}}, \frac{-d}{2c}\right), A_3\left(-\sqrt[4]{\frac{-b(n-4)}{an}}, \frac{-d}{2c}\right)$$

but

$$U(A_1) = -\frac{1}{4c}\left(d^2 - 4ch\right) \neq 0$$

because $h \neq \dfrac{d^2}{4c}$ for $n \neq 5$, and when $n = 5$ the possible critical points on Γ are A_2 and A_3.

$$U(A_2) = U(A_3) = -4\left(\frac{a}{n-4}\right)^{1-\frac{n}{4}}\left(\frac{-b}{n}\right)^{\frac{n}{4}} - \frac{d^2}{4c} + h = -\alpha_n - \frac{d^2}{4c} + h \neq 0$$

because $h \neq \dfrac{d^2}{4c} + \alpha_n$ then the curve is nonsingular.

We mark that, when $n = 4$, The only possible critical point on Γ is A_1.

Now we prove that the curveΓ contains one or two ovals. ■

We consider the equation (2.3) as

$$cy^2 + dy + \left(ax^n + bx^{n-4} + h\right) = 0 \ \ , n \geq 4 \tag{3.3}$$

the discriminant

$$\Delta = -4c\left(ax^n + bx^{n-4} + h\right) + d^2$$

$$\frac{d\left(\Delta(x)\right)}{dx} = -4cx^{n-5}\left(anx^4 + b\left(n-4\right)\right)$$

To study the sign of the derivative we distinguish two cases.

1) n is odd number

$$\frac{d\left(\Delta(x)\right)}{dx}$$

is positive if

$$x \in \left]-\sqrt[4]{\frac{-b(n-4)}{an}}, \sqrt[4]{\frac{-b(n-4)}{an}}\right[,$$

and negative if

$$x \in \left]-\infty, -\sqrt[4]{\frac{-b(n-4)}{an}}\right[\cup \left]\sqrt[4]{\frac{-b(n-4)}{an}}, +\infty\right[$$

$$\Delta\left(-\sqrt[4]{\frac{-b(n-4)}{an}}\right) = -4c\left(\alpha_n + h\right) + d^2$$

and

$$\Delta\left(\sqrt[4]{\frac{-b(n-4)}{an}}\right) = -4c\left(-\alpha_n + h\right) + d^2$$

if

$$h \in \left]\frac{d^2}{4c} - \alpha_n, \frac{d^2}{4c}\right[\cup \left]\frac{d^2}{4c}, \frac{d^2}{4c} + \alpha_n\right[,$$

$$-4c\left(\alpha_n + h\right) + d^2 < 0$$

and $-4c\left(-\alpha_n + h\right) + d^2 > 0$.

By applying the intermediate value theorem, we note that there are two solutions of equation $\Delta(x) = 0$

$$x_1 \in \left]-\sqrt[4]{\frac{-b(n-4)}{an}}, \sqrt[4]{\frac{-b(n-4)}{an}}\right[,$$

$$x_2 \in \left]\sqrt[4]{\frac{-b(n-4)}{an}}, +\infty\right[$$

and we conclude that $\forall x \in]x_1, x_2[$, $\Delta(x) > 0$, then the equation (3.3) has two solutions that depend on y

$$y_1 = \frac{-d - \sqrt{-4c\left(ax^n + bx^{n-4} + h\right) + d^2}}{2c},$$

$$y_2 = \frac{-d + \sqrt{-4c\left(ax^n + bx^{n-4} + h\right) + d^2}}{2c}$$

note that $\Delta(x_1) = \Delta(x_2) = 0$ and $y_1(x_1) = y_2(x_1)$, $y_1(x_2) = y_2(x_2)$ on the other hand $x \to y_1(x)$ is a decreasing function for $x \in \left]x_1, \sqrt[4]{\frac{-b(n-4)}{an}}\right[$ and an increasing function for $x \in \left]\sqrt[4]{\frac{-b(n-4)}{an}}, x_2\right[$. $x \to y_2(x)$ is an increasing function for $x \in \left]x_1, \sqrt[4]{\frac{-b(n-4)}{an}}\right[$ and a decreasing function for $x \in \left]\sqrt[4]{\frac{-b(n-4)}{an}}, x_2\right[$ so for $x \in [x_1, x_2]$ the curve Γ is formed by one oval.

Note that if $h < \frac{d^2}{4c} - \alpha_n$ or $h \geq \frac{d^2}{4c} + \alpha_n$ the curve can not presented an oval, and when $h = \frac{d^2}{4c} - \alpha_n$, the curve Γ resembles a form of strophoid.

2) n is even number

$$\frac{d\left(\Delta(x)\right)}{dx}$$

is positive if

$$x \in \left]-\infty, -\sqrt[4]{\frac{-b(n-4)}{an}}\right[\cup \left]0, \sqrt[4]{\frac{-b(n-4)}{an}}\right[,$$

and negative if

$$x \in \left]-\sqrt[4]{\frac{-b(n-4)}{an}}, 0\right[\cup \left]\sqrt[4]{\frac{-b(n-4)}{an}}, +\infty\right[$$

$$\Delta\left(-\sqrt[4]{\frac{-b(n-4)}{an}}\right) = \Delta\left(\sqrt[4]{\frac{-b(n-4)}{an}}\right) = -4c\left(-\alpha + h\right) + d^2,$$

$$\Delta\left(0\right) = -4ch + d^2$$

if

$$\frac{d^2}{4c} < h < \frac{d^2}{4c} + \alpha, -4c\left(-\alpha_n + h\right) + d^2 > 0$$

and $-4ch + d^2 < 0$. By applying the intermediate value theorem, we conclude that there are four solutions of equation $\Delta(x) = 0$

$$x_1 \in \left]-\infty, -\sqrt[4]{\frac{-b(n-4)}{an}}\right[,$$

$$x_2 \in \left]-\sqrt[4]{\frac{-b(n-4)}{an}}, 0\right[,$$

$$x_3 \in \left]0, \sqrt[4]{\frac{-b(n-4)}{an}}\right[,$$

$$x_4 \in \left]\sqrt[4]{\frac{-b(n-4)}{an}}, +\infty\right[$$

and we conclude that

$$\forall x \in \left]x_1, x_2\right[\cup \left]x_3, x_4\right[, \Delta(x) > 0,$$

then the equation (3.3) has two solutions that depend on y

$$y_1 = \frac{-d - \sqrt{-4c\left(ax^n + bx^{n-4} + h\right) + d^2}}{2c},$$

$$y_2 = \frac{-d + \sqrt{-4c\left(ax^n + bx^{n-4} + h\right) + d^2}}{2c}$$

note that

$$y_1(x_1) = y_2(x_1), y_1(x_2) = y_2(x_2),$$

$$y_1(x_3) = y_2(x_3), y_1(x_4) = y_2(x_4)$$

and $x \to y_1(x)$ is a decreasing function for

$$x \in \left]x_1, -\sqrt[4]{\frac{-b(n-4)}{an}}\right[\cup \left]x_3, \sqrt[4]{\frac{-b(n-4)}{an}}\right[$$

and an increasing function for

$$x \in \left]-\sqrt[4]{\frac{-b(n-4)}{an}}, x_2\right[\cup \left]\sqrt[4]{\frac{-b(n-4)}{an}}, x_4\right[$$

$x \to y_2(x)$ is an increasing function for

$$x \in \left]x_1, -\sqrt[4]{\frac{-b(n-4)}{an}}\right[\cup \left]x_3, \sqrt[4]{\frac{-b(n-4)}{an}}\right[$$

and a decreasing function for

$$x \in \left]-\sqrt[4]{\frac{-b(n-4)}{an}}, x_2\right[\cup \left]\sqrt[4]{\frac{-b(n-4)}{an}}, x_4\right[$$

so the curve Γ is composed by two ovals, one for $x \in [x_1, x_2]$ and the other for $x \in [x_3, x_4]$.

If

$$h < \frac{d^2}{4c}, -4c(-\alpha_n + h) + d^2 > 0$$

and $-4ch + d^2 > 0$ in this case the equation $\Delta(x) = 0$ admits two solutions

$$x_1 \in \left]-\infty, -\sqrt[4]{\frac{-b(n-4)}{an}}\right[,$$

$$x_2 \in \left]\sqrt[4]{\frac{-b(n-4)}{an}}, +\infty\right[$$

$$\forall x \in]x_1, x_2[, \Delta(x) > 0,$$

the equation (7) has two solutions that depend on y and we conclude as before that Γ is formed by one oval.

Note that if $h \geq \frac{d^2}{4c} + \alpha_n$ the curve Γ can not presented an oval and when $h = \frac{d^2}{4c}$, the curve Γ has a form of Lemniscate.

We mark the special case, when $n = 4$, the curve Γ present one oval if $h < \frac{d^2}{4c}$, and when $h \geq \frac{d^2}{4c}$ it can not presented an oval.

ii) Γ is an invariant curve of the system (3.1) and $\int_0^T K(x(t), y(t))dt$ is nonzero.

$$\frac{dU}{dt} = U_x \dot{x} + U_y \dot{y} = U\left(VU_x + WU_y\right)$$

The cofactor is $K(x, y) = V(x, y)U_x(x, y) + W(x, y)U_y(x, y)$.

When $n = 4$

$$K(x, y) = 4ax^3 V(x, y) + (d + 2cy)\, W(x, y)$$

When $n > 4$

$$K(x, y) = \left(x^{n-5}\left(anx^4 + b(n-4)\right)\right) V(x, y) + (d + 2cy)\, W(x, y)$$

It remains to prove that $\int_0^T K(x, y)dt \neq 0$

$$\int_0^T K(x(t), y(t))dt = \int_\Gamma \frac{V(x, y)U_x(x, y)}{U_x(x, y)} dy + \int_\Gamma \frac{W(x, y)U_y(x, y)}{-U_y(x, y)} dx$$
$$= \int_\Gamma V(x, y)dy - \int_\Gamma W(x, y)dx$$

by applying the GREEN formula

$$\int_\Gamma V(x, y)dy - \int_\Gamma W(x, y)dx = \int\int_{\int(\Gamma)} \left(\frac{\partial V(x, y)}{\partial x} + \frac{\partial W(x, y)}{\partial y}\right) dxdy$$

$int(\Gamma)$ denotes the interior of Γ and as

$$\forall (x, y) \in \int (\Gamma), \frac{\partial V(x, y)}{\partial x} + \frac{\partial W(x, y)}{\partial y}$$

is positive, then $\int_0^T K(x(t), y(t))dt$ is nonzero.

4. Examples

Example 4.1. Quintic system with one limit cycle. $n = 4, a = 1, b = 0, c = 1, d = -4, h = 3, V(x, y) = x - y - 1, W(x, y) = x + y + 1$ the system

$$\begin{cases} \dot{x} = x^5 - x^4 y - x^4 + xy^2 - 4xy + 3x - y^3 + 3y^2 - y + 1 \\ \dot{y} = x^5 + x^4 y + x^4 + 4x^3 + xy^2 - 4xy + 3x + y^3 - 3y^2 - y + 3 \end{cases}$$

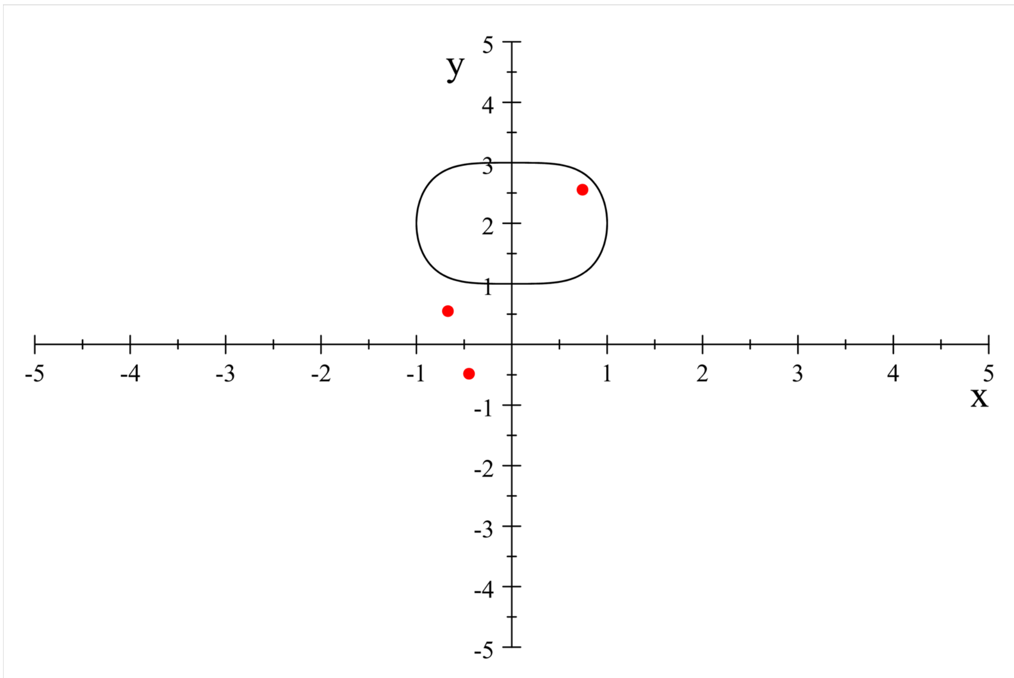

Figure 1:

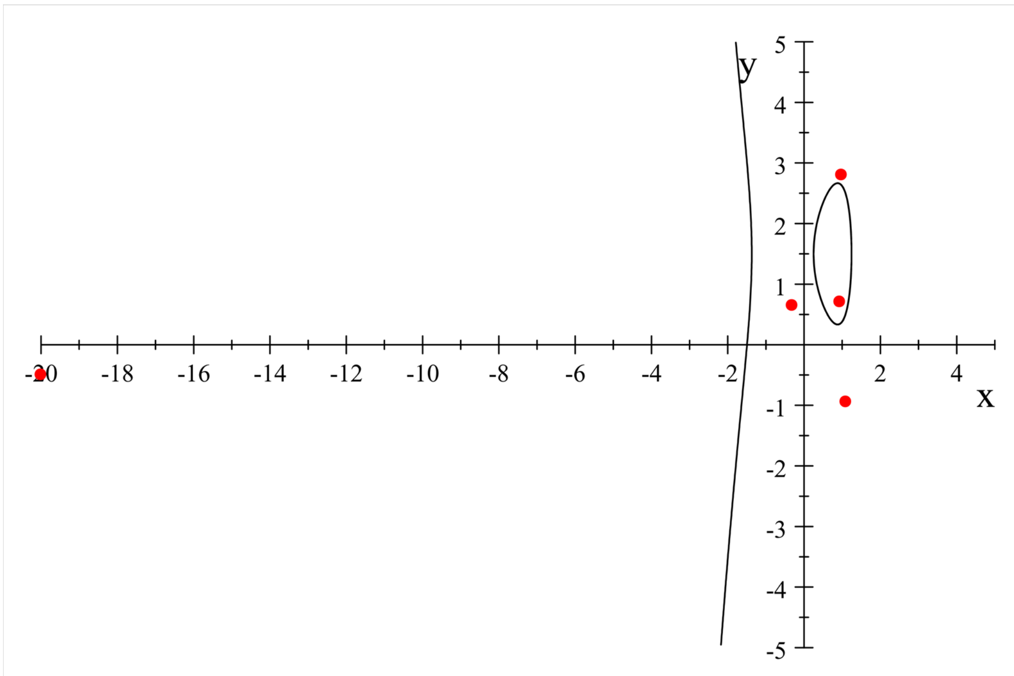

Figure 2:

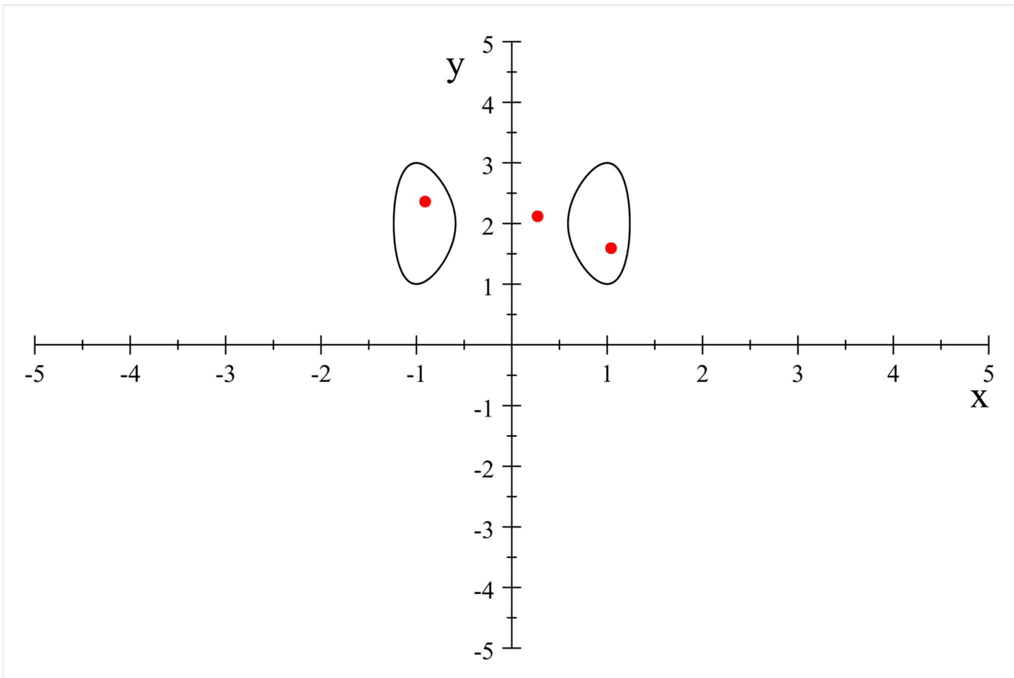

Figure 3:

admits one limit cycle represented by the curve $x^4 + y^2 - 4y + 3 = 0$. This curve is nonsingular, the system admits three singular points, one is a saddle points, one is a stable focus and one is an instable focus, the limit cycle encloses a stable focus (figure 1).

Example 4.2. Septic system with one limit cycle. $n = 5, a = 1$, $b = -3$, $c = 1$, $d = -3$, $h = 3$, $V(x, y) = 2xy + x$, $W(x, y) = -y^2 + y + 1$. The system

$$\begin{cases} \dot{x} = 2x^6y + x^6 - 6x^2y - 3x^2 + 2xy^3 - 5xy^2 + 3xy + 3x - 2y + 3 \\ \dot{y} = -x^5y^2 + x^5y + x^5 + 5x^4 + 3xy^2 - 3xy - 3x - y^4 + 4y^3 - 5y^2 \end{cases}$$

admits one limit cycle represented by the curve $x^5 - 3x + y^2 - 3y + 3 = 0$. This curve is nonsingular, the system admits five singular points, four are saddle points, one is stable focus and it is inside the limit cycle (figure 2).

Example 4.3. Septic system with two limit cycles $n = 6$, $a = 1$, $b = -3$, $c = 1$, $d = -4$, $h = 5$, $V(x, y) = x$, $W(x, y) = y$. The system

$$\begin{cases} \dot{x} = x^7 - 3x^3 + xy^2 - 4xy + 5x - 2y + 4 \\ \dot{y} = x^6y + 6x^5 - 3x^2y - 6x + y^3 - 4y^2 + 5y \end{cases}$$

admits two limit cycles represented by the curve $x^6 - 3x^2 + y^2 - 4y + 5 = 0$. This curve is nonsingular, the system admits three singular points, two are stable focus, inside the limit cycles and one is a saddle point, it is outside (figure 3).

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