Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 4 (2015), pp. 1747-1754

http://www.ripublication.com

Degenerate Daehee polynomials associated with p-adic invariant integral on $\mathbb{Z}_p$

JongKyum Kwon

Department of Mathematics,
Kyungpook National University,
Taegu 702-701, Republic of Korea.

Seog-Hoon Rim

Department of Mathematics Education,
Kyungpook National University,
Taegu 702-701, Republic of Korea.
E-mail: shrim@knu.ac.kr

Jin-Woo Park[1]

Department of Mathematics Education,
Daegu University, Gyeongsan-si,
Gyeongsangbuk-do, 712-714, Republic of Korea.
E-mail: a0417001@knu.ac.kr

Abstract

In this paper, we consider the degenerate Daehee polynomials which are derived from p-adic invariant integral on $\mathbb{Z}_p$ and investigate some properties of those polynomials.

AMS subject classification: 05A10, 05A19.
Keywords: Bernoulli polynomial, Daehee polynomials, degenerate Daehee polynomials.

[1]Corresponding author.

1. Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is normally defined by $|p|_p = \frac{1}{p}$. Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. Then the *p-adic invariant integral on* $\mathbb{Z}_p$ is defined as

$$\begin{aligned}\int_{\mathbb{Z}_p} f(x)d\mu_0(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)d\mu_0\left(x+p^N\mathbb{Z}_p\right)\\ &= \lim_{N\to\infty} \frac{1}{p^N} \sum_{n=0}^{p^N-1} f(x), \text{ (see [11, 12, 15]).}\end{aligned} \tag{1.1}$$

From (1.1), we have

$$I_0(f_1) - I_0(f) = f^{'}(0). \tag{1.2}$$

As is well known, the *Bernoulli polynomials* are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \text{ (see [1-2, 4-12]).} \tag{1.3}$$

When $x = 0$, $B_n = B_n(0)$, $(n \geq 0)$, are called the *ordinary Bernoulli numbers*.

In [2], L. Carlitz consider the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{x}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(x|\lambda)\frac{t^n}{n!}. \tag{1.4}$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(0|\lambda)$ are called the *degenerate Bernoulli numbers*. Note that $\lim_{\lambda\to 0} \beta_n(\lambda) = B_n$.

The Daehee polynomials are defined by

$$D_n(x) = \int_{\mathbb{Z}_p} (x+y)_n d\mu_0(y), \ (n \geq 0), \text{ (see [6, 9, 10]).} \tag{1.5}$$

From (1.2) and (1.5), we can derive the generating function to be

$$\begin{aligned}\sum_{n=0}^{\infty} D_n^{(r)}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{x_1+\cdots+x_r+x} d\mu_0(x_1)\cdots d\mu_0(x_r)\\ &= \left(\frac{\log(1+t)}{t}\right)^r (1+t)^x,\end{aligned} \tag{1.6}$$

(see [6, 9]).

By (1.3) and (1.6), it is not difficult to show that

$$\left(\frac{\log(1+t)}{t}\right)^r (1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n+r+1)}(x)\frac{t^n}{n!}, \text{ (see [6]).} \tag{1.7}$$

It is well known that

$$e^t = \lim_{\lambda \to 0}(1+\lambda t)^{\frac{1}{\lambda}}.$$

Now, the function $(1+\lambda t)^{\frac{1}{\lambda}}$ is called the *degenerate function of e^t*.

$$t = \log e^t \quad \text{———} \quad \int_{\mathbb{Z}_p}(1+t)^x d\mu_0(x) = \frac{t}{\log(1+t)}$$

(Daehee number)

$$\log\left((1+\lambda t)^{\frac{1}{\lambda}}\right) \quad \text{———} \quad \int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x d\mu_0(x) = ?$$

(degenerate of t)

In this paper, we consider the degenerate Daehee numbers and polynomials which are derived from p-adic invariant integral on $\mathbb{Z}_p$ and investigate some properties of those polynomials.

2. Degenerate Daehee polynomials

Let us assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$. We define the *degenerate Daehee polynomials* by the generating function as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x+y} d\mu_0(y) \\ &= \frac{\log\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x.\end{aligned} \tag{2.1}$$

When $x = 0$, $D_{n,\lambda} = D_{n,\lambda}(0)$ are called the *n-th degenerate Daehee numbers*.

It is well-known fact that the generating function of the Stirling number of the first kind is given by

$$(\log(1+t))^m = m!\sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!}, \text{ (see [3, 8, 14]),} \tag{2.2}$$

and the Stirling number of the second kind is defined by the generating function to be

$$(e^t - 1)^n = \sum_{m=n}^{\infty} S_2(m, n)\frac{t^m}{m!}, \text{ (see [3, 14]).}$$

By (2.1) and (2.2), we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p} \left(1 + \log(1 + \lambda t)^{\frac{1}{\lambda}}\right)^{x+y} d\mu_0(y) &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x+y}{n} \lambda^{-n} \left(\log(1 + \lambda t)\right)^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) \int_{\mathbb{Z}_p} (x + y)_l d\mu_0(y) \right) \frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

Thus, by (2.1) and (2.3), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$D_{n,\lambda}(x) = \sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) \int_{\mathbb{Z}_p} (x + y)_l d\mu_0(y).$$

By (1.5) and (1.6), we can note that

$$\begin{aligned}\int_{\mathbb{Z}_p} (1 + t)^{x+y} d\mu_0(y) &= \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} (x + y)_n d\mu_0(y) \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

Therefore, by Theorem 2.1 and (2.5), we obtain the following corollary.

Corollary 2.2. For $n \geq 0$, we have

$$D_{n,\lambda}(x) = \sum_{l=0}^{n} \lambda^{n-l} S_1(n, l) D_l(x).$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (2.1), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} (1 + t)^{x+y} d\mu_0(y) &= \sum_{n=0}^{\infty} D_{n,\lambda}(x) \frac{1}{n!} \lambda^{-n} \left(e^{\lambda t} - 1\right)^n \\ &= \sum_{n=0}^{\infty} D_{n,\lambda}(x) \lambda^{-n} \sum_{m=n}^{\infty} S_2(m, n) \frac{(\lambda t)^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} D_{m,\lambda}(x) \lambda^{n-m} S_2(n, m) \right) \frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

By (1.6) and (2.5), we obtain the following corollary.

Corollary 2.3. For $n \geq 0$, we have

$$D_n(x) = \sum_{m=0}^{n} D_{m,\lambda}(x)\lambda^{n-m} S_2(n, m).$$

By (1.7), we can derive the following equations easily:

$$\begin{aligned}
&\frac{\log\left(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\right)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\left(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x \\
&= \sum_{n=0}^{\infty} B_n^{(n+2)}(x+1)\frac{1}{n!}\left(\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^n \\
&= \sum_{n=0}^{\infty} B_n^{(n+2)}(x+1)\lambda^{-n}\sum_{l=n}^{\infty} S_1(l,n)\frac{(\lambda t)^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_m^{(m+2)}(x+1)\lambda^{n-m} S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.6}$$

By (2.1) and (2.6), we obtain the following theorem.

Theorem 2.4. For $n \geq 0$, we have

$$D_{n,\lambda}(x) = \sum_{m=0}^{n} B_m^{(m+2)}(x+1)\lambda^{n-m} S_1(n, m).$$

We can observe that

$$\begin{aligned}
&\frac{\log\left(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\right)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\left(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x \\
&= \left(\sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\binom{x}{n}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)\right)^x \\
&= \left(\sum_{n=0}^{\infty} D_{n,\lambda}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\lambda^{-k}(x)_k S_1(m,k)\right)\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m} D_{n-m,\lambda}\binom{x}{l}\binom{n}{m} l!\lambda^{-l} S_1(m,l)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.7}$$

Thus, by (2.1) and (2.7), we obtain the following theorem.

Theorem 2.5. For $n \geq 0$, we have

$$D_{n,\lambda}(x) = \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{x}{l} \binom{n}{m} l! \lambda^{-l} S_1(m,l) D_{n-m,\lambda}. \tag{2.8}$$

It is well-known that the *Daehee polynomials of the second kind* are defined by the generating function to be

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+t)^{x-y} d\mu_0(y) &= \frac{\log(1+t)}{1-(1+t)^{-1}} (1+t)^x \\ &= \sum_{n=0}^{\infty} \widehat{D}_n(x) \frac{t^n}{n!}, \end{aligned} \tag{2.9}$$

(see [6, 9, 10]).

From now on, we consider the *degenerate Daehee polynomials of the second kind* which are defined by the generating function to be

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{D}_{n,\lambda}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} \left(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x-y} d\mu_0(y) \\ &= \frac{\log\left(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\right)}{1 - \left(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{-1}} \left(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x}. \end{aligned} \tag{2.10}$$

By (2.10), we observe that

$$\begin{aligned} &\int_{\mathbb{Z}_p} \left(1 + \log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x-y} d\mu_0(y) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x-y}{n} \left(\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^n d\mu_0(y) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x-y}{n} d\mu_0(y) n! \sum_{m=n}^{\infty} S_1(m,n) \frac{(\lambda t)^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \lambda^n S_1(n,m) \int_{\mathbb{Z}_p} (x-y)_m d\mu_0(y) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.11}$$

By (2.10) and (2.11), we obtain the following theorem.

Theorem 2.6. For $n \geq 0$, we have

$$\widehat{D}_{n,\lambda}(x) = \sum_{m=0}^{n} \lambda^n S_1(n,m) \widehat{D}_m(x).$$

By (2.10), we have

$$
\begin{aligned}
&\frac{\log\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^{x} \\
&=\sum_{n=0}^{\infty} B_n^{(n+2)}(x+2)\frac{\left(\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^n}{n!} \\
&=\sum_{n=0}^{\infty} B_n^{(n+2)}(x+2)\lambda^{-n}\sum_{l=n}^{\infty} S_1(l,n)\frac{(\lambda t)^l}{l!} \\
&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_m^{(m+2)}(x+2)\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.12}
$$

Therefore, by (2.12), we obtain the following theorem.

Theorem 2.7. For $n \geq 0$, we have

$$
\widehat{D}_{n,\lambda}(x)=\sum_{m=0}^{n} B_m^{(m+2)}(x+2)\lambda^{n-m}S_1(n,m).
$$

From (2.10) and (2.11), we have

$$
\begin{aligned}
\widehat{D}_{n,\lambda}(x) &= \sum_{m=0}^{n}\lambda^n S_1(n,m)\int_{\mathbb{Z}_p}(x-y)_m d\mu_0(y) \\
&= \sum_{m=0}^{n}\lambda^n S_1(n,m)(-1)^m\int_{\mathbb{Z}_p}(y-x+m-1)_m d\mu_0(y) \\
&= \sum_{m=0}^{n}\lambda^n S_1(n,m)(-1)^m m!\int_{\mathbb{Z}_p}\binom{y-x+m-1}{m}d\mu_0(y) \\
&= \sum_{m=0}^{n}\lambda^n S_1(n,m)(-1)^m m!\int_{\mathbb{Z}_p}\sum_{l=0}^{m}\binom{m-1}{m-l}\binom{y-x}{l}d\mu_0(y) \\
&= \sum_{m=0}^{n}\sum_{l=0}^{m}\lambda^n S_1(n,m)(-1)^m m!\binom{m-1}{m-l}\frac{1}{l!}\int_{\mathbb{Z}_p}(y-x)_l d\mu_0(y) \\
&= \sum_{m=0}^{n}\sum_{l=1}^{m}\lambda^n S_1(n,m)(-1)^m m!\binom{m-1}{m-l}\frac{1}{l!}D_l(-x).
\end{aligned} \tag{2.13}
$$

By (2.13), we obtain the following theorem.

Theorem 2.8. For $n \geq 0$, we have

$$\widehat{D}_{n,\lambda}(x) = \sum_{m=0}^{n} \sum_{l=1}^{m} \frac{(-1)_m}{l!} \lambda^n S_1(n,m) \binom{m-1}{m-l} D_l(-x).$$

References

[1] A. Bayad and T. Kim, *Identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **20** (2010), no. 2, 247–253.

[2] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., **15** (1979), 51–88.

[3] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[4] D. Ding and J. Yang, *Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), **20** (2010), no. 1, 7–21.

[5] S. Gaboury, R. Tremblay and B.-J. Fugere, *Some explicit formulas for certain new classes of Bernoulli, Euler and Genocchi polynomials*, Proc. Jangjeon Math. Soc., **17** (2014), no. 1, 115–123.

[6] D. S. Kim and T. Kim, *Daehee numbers and polynomials*, Appl. Math. Sci., **7** (2013), no. 117–120, 5969–5976.

[7] D. S. Kim, T. Kim and D.V. Dolgy, *A note on degenerate Bernoulli numbers and polynomials associated with p-adic invariant integral on* $\mathbb{Z}_p$, Appl. Math. Comput., **259** (2015), 198–204.

[8] D. S. Kim, T. Kim and J. J. Seo, *Higher-order Daehee polynomials of the first kind with umbral calculus*, Adv. Stud. Contemp. Math., **24** (2014), no. 1, 5–18.

[9] D. S. Kim, T. Kim, S. H. Lee and J. J. Seo, *Higher-order Daehee numbers and polynomials*, Int. J. Math. Anal., **24** (2014), no. 1, 5–18.

[10] D. S. Kim, T. Kim, H. I. Kwon and J. J. Seo, *Daehee Polynomials with q-Parameter*, Adv. Studies Theor. Phys., **8** (2014), no. 13, 561–569.

[11] T. Kim, *Barnes' type multiple degenerate Bernoulli and Euler polynomials*, Appl. Math. Comput., **258** (2015), 556–564.

[12] T. Kim, *Symmetry p-adic invariant integral on* $\mathbb{Z}_p$ *for Bernoulli and Euler polynomials*, J. Difference Equ. Appl., **14** (2008), no. 12, 1267–1277.

[13] T. Kim, *Multiple p-adic L-function*, Russ. J. Math. Phys., **13** (2006), no. 2, 151–157.

[14] S. Roman, *The umbral calculus*, Dover, New York, 2005.

[15] L. C. Washington, *Introduction to cyclotomic fields, Second edition. Graduate Texts in Mathematics, 83*, Springer-Verlag, New York, 1997.