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On the λ-Bernoulli and Euler polynomials

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Abstract

In this paper, we consider the λ-analogue of Bernoulli and Euler polynomials, and give some identities of those polynomials.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $f(x)$ be a uniformly differentiable function on $\mathbb{Z}_p$. The *p-adic bosonic integral on $\mathbb{Z}_p$* is defined by

$$I_0(f) = \int_{\mathbb{Z}_p} f(x)d\mu_0(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \text{ (see [2, 3])}, \tag{1.1}$$

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and the *fermionic integral on* $\mathbb{Z}_p$ is defined by

$$I_{-1}(f)=\int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x)=\lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x)(-1)^x,\ \text{(see [2, 3])}. \tag{1.2}$$

From (1.1) and (1.2), we have

$$I_0(f_1)-I_0(f)=f^{'}(0), \tag{1.3}$$

and

$$I_{-1}(f_1)+I_{-1}(f)=2f(0),\ \text{where } f_1(x)=f(x+1). \tag{1.4}$$

By using interative method, we get

$$I_0(f_n)-I_0(f)=\sum_{l=0}^{n-1} f^{'}(l), \tag{1.5}$$

and

$$I_{-1}(f_n)+(-1)^{n-1}I_{-1}(f)=2\sum_{l=0}^{n-1} f(l)(-1)^{n-1-l}, \tag{1.6}$$

where $n\in\mathbb{N}$ and $f_n(x)=f(x+n)$ (see [2]).

From (1.3) and (1.4), we note that

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y)=\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \tag{1.7}$$

and

$$\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_0(y)=\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{1.8}$$

In this paper, we consider the λ-analogue of Bernoulli and Euler polynomials, and give some identities of those polynomials.

2. λ-Bernoulli and Euler polynomials

In this section, we assume that $\lambda, t\in\mathbb{C}_p$ such that $|\lambda t|_p<p^{-\frac{1}{p-1}}$.

Let us consider the following integral:

$$\int_{\mathbb{Z}_p} e^{\frac{y}{\lambda}\log(1+\lambda t)+xt}d\mu_0(y).$$

From (1.3), we can derive the following equation:

$$\frac{t}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\int_{\mathbb{Z}_p} e^{\frac{y}{\lambda}\log(1+\lambda t)+xt}d\mu_0(y)=\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}e^{xt}. \tag{2.1}$$

Now, we define the λ-*Bernoulli polynomials* as follows:

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}e^{xt}=\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.2}$$

Note that $B_{n,\lambda}(x)$ is Apell polynomials for the part $\left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}-1}{t}, t\right)$, (see [1, 5]).

We observe that

$$\begin{aligned}&\frac{t}{\log(1+\lambda t)^{\frac{1}{\lambda}}}e^{xt}\\&\quad=\left(\sum_{l=0}^{\infty}\frac{\lambda^l}{l!}t^lB_l^{(l)}(1)\right)\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}t^m\right)\\&\quad=\sum_{k=0}^{\infty}\left(\sum_{l=0}^{k}\binom{k}{l}\lambda^lB_l^{(l)}(1)x^{k-l}\right)\frac{t^k}{k!},\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}_p}e^{\frac{y}{\lambda}\log(1+\lambda t)}d\mu_0(y)\\&\quad=\sum_{m=0}^{\infty}\lambda^{-m}\int_{\mathbb{Z}_p}y^md\mu_0(y)\frac{1}{m!}\left(\log(1+\lambda t)\right)^m\\&\quad=\sum_{m=0}^{\infty}\lambda^{-m}B_m\sum_{j=m}^{\infty}S_1(j,m)\frac{\lambda^jt^j}{j!}\\&\quad=\sum_{j=0}^{\infty}\left(\sum_{m=0}^{j}\lambda^{j-m}B_mS_1(j,m)\right)\frac{t^j}{j!},\end{aligned} \tag{2.4}$$

where $S_1(j,m)$ is the Stirling number of the first kind.

From (2.3) and (2.4), we have

$$\begin{aligned}&\frac{t}{\log(1+\lambda t)^{\frac{1}{\lambda}}}\int_{\mathbb{Z}_p}e^{\frac{y}{\lambda}\log(1+\lambda t)+xt}d\mu_0(y)\\&=\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{m=0}^{n-k}\binom{n}{k}\binom{k}{l}\lambda^{l+n-k-m}B_l^{(l)}(1)B_mS_1(n-k,m)x^{k-l}\right\}\frac{t^n}{n!}.\end{aligned} \tag{2.5}$$

Therefore, by (2.2) and (2.5), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$,

$$B_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{m=0}^{n-k}\binom{n}{k}\binom{k}{l}\lambda^{l+n-k-m}B_l^{(l)}(1)B_m S_1(n-k,m)x^{k-l}.$$

When $x = 0$, $B_{n,\lambda} = B_{n,\lambda}(0)$ are called the λ-*Bernoulli numbers*.

Note that $\lim_{\lambda\to 0} B_{n,\lambda}(x) = B_n(x)$, $(n \geq 0)$. From (2.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}e^{xt} = \left(\sum_{n=0}^{\infty} B_{m,\lambda}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}t^m\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}B_{m,\lambda}x^{n-m}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

By comparing the coefficients of the both sides of (2.6), we get

$$\begin{aligned}B_{n,\lambda}(x) &= \sum_{m=0}^{n}\binom{n}{m}B_{m,\lambda}x^{n-m}\\ &= (B_\lambda + x)^n\,,\ (n \geq 0),\end{aligned} \tag{2.7}$$

with the usual convention about replacing B_λ^n by $B_{n,\lambda}$. From (2.7), we have

$$\begin{aligned}\frac{d}{dx}B_{n,\lambda}(x) &= \frac{d}{dx}(B_\lambda + x)^n = n(B_\lambda + x)^{n-1}\\ &= nB_{n-1,\lambda}(x),\ (n \geq 1).\end{aligned} \tag{2.8}$$

Therefore, by (2.7) and (2.8), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$B_{n,\lambda}(x) = \sum_{l=0}^{n}\binom{n}{l}B_{l,\lambda}x^{n-l} = (B_\lambda + x)^n.$$

In particular,

$$\frac{d}{dx}B_{n,\lambda}(x) = nB_{n-1,\lambda}(x).$$

It is not difficult to show that

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1} = \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}\sum_{l=0}^{d-1}(1+\lambda t)^{\frac{l}{\lambda}},\ (d \in \mathbb{N}). \tag{2.9}$$

The Carlitz's degenerate Bernoulli polynomials are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\beta_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.10}$$

From (2.9) and (2.10), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}e^{xt} \\
&= \frac{1}{d}\sum_{l=0}^{d-1}\frac{dt}{\left(1+\frac{\lambda}{d}dt\right)^{\frac{d}{\lambda}}-1}\left(1+\frac{\lambda}{d}dt\right)^{\frac{l}{d}\cdot\frac{d}{\lambda}} e^{xt} \\
&= \sum_{k=0}^{\infty}\left(d^{k-1}\sum_{l=0}^{d-1}\beta_{k,\frac{\lambda}{d}}\left(\frac{l}{d}\right)\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\frac{x^m}{m!}t^m\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} d^{k-1}\binom{n}{k}\sum_{l=0}^{d-1}\beta_{k,\frac{\lambda}{d}}\left(\frac{l}{d}\right)x^{n-k}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.11}$$

Therefore, by (2.11), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, we have

$$B_{n,\lambda}(x) = \sum_{k=0}^{n} d^{k-1}\binom{n}{k}\sum_{l=0}^{d-1}\beta_{n,\frac{\lambda}{d}}\left(\frac{l}{d}\right)x^{n-k}.$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.1), we get

$$\begin{aligned}
&\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_0(y) \\
&\quad = \sum_{m=0}^{\infty} B_{m,\lambda}(x)\frac{1}{m!}\left(\frac{1}{\lambda}(e^{\lambda t}-1)\right)^m \\
&\quad = \sum_{m=0}^{\infty} B_{m,\lambda}(x)\lambda^{-m}\sum_{n=m}^{\infty} S_2(n,m)\frac{\lambda^n t^n}{n!} \\
&\quad = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_{m,\lambda}(x)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!},
\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}
\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_0(y) &= \frac{t}{e^t-1}e^{xt} \\
&= \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},
\end{aligned} \tag{2.13}$$

where $S_2(n, m)$ is the Stirling number of the second kind.

Therefore, by (2.12) and (2.13), we obtain the following theorem.

Theorem 2.4. For $n \geq 0$, we have

$$B_n(x) = \sum_{m=0}^{n} B_{m,\lambda}(x)\lambda^{n-m} S_2(n,m).$$

By replacing t by $\frac{1}{\lambda}\log(1+\lambda t)$ in (1.8), we get

$$\begin{aligned}
\sum_{m=0}^{\infty} B_m(x)\frac{1}{m!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m &= \int_{\mathbb{Z}_p} e^{(x+y)\frac{1}{\lambda}\log(1+\lambda t)}d\mu_0(y) \\
&= \int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+y}{\lambda}}d\mu_0(y) = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \\
&= \frac{\log(1+\lambda t)}{\lambda t}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}\right) \qquad (2.14) \\
&= \left(\sum_{l=0}^{\infty}\lambda^l D_l \frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\beta_{m,\lambda}(x)\frac{t^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\lambda^l D_l \beta_{n-l,\lambda}(x)\right)\frac{t^n}{n!},
\end{aligned}$$

where D_n is the n-th Daehee numbers.

On the other hand,

$$\begin{aligned}
&\sum_{m=0}^{\infty} B_m(x)\frac{1}{\lambda^m}\frac{1}{m!}(\log(1+\lambda t))^m \\
&= \sum_{m=0}^{\infty} B_m(x)\frac{1}{\lambda^m}\sum_{n=m}^{\infty} S_1(n,m)\frac{\lambda^n t^n}{n!} \qquad (2.15) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} B_m(x)\lambda^{n-m}S_1(n,m)\right)\frac{t^n}{n!}.
\end{aligned}$$

Therefore, by (2.14) and (2.15), we get the following theorem.

Theorem 2.5. For $n \geq 0$, we have

$$\sum_{n=0}^{n}\binom{n}{l}\lambda^l D_l \beta_{n-l,\lambda}(x) = \sum_{m=0}^{n} B_m(x)\lambda^{n-m}S_1(n,m).$$

3. λ-Euler polynomials

For $\lambda, t \in \mathbb{C}_p$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$, we have

$$\int_{\mathbb{Z}_p} e^{\frac{y}{\lambda}\log(1+\lambda t)+xt} d\mu_{-1}(y) = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} e^{xt}. \tag{3.1}$$

Now, we define the degenerate λ-Euler polynomials which are given by the generating function to be

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} = \sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!}. \tag{3.2}$$

When $x = 0$, $E_{n,\lambda} = E_{n,\lambda}(0)$ are called the *degenerate λ-Euler numbers*.

From (3.2), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} e^{xt} \\ &= \left(\sum_{l=0}^{\infty} E_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} x^m \frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} E_{l,\lambda}x^{n-l}\right)\frac{t^n}{n!}. \end{aligned} \tag{3.3}$$

Thus, by (3.3), we get

$$E_{n,\lambda}(x) = \sum_{l=0}^{n}\binom{n}{l}E_{l,\lambda}x^{n-l} = (E_\lambda + x)^n, \tag{3.4}$$

with the usual convention about replacing E_λ^n by $E_{n,\lambda}$. By (3.4), we get

$$\begin{aligned} \frac{d}{dx}E_{n,\lambda}(x) &= \frac{d}{dx}(E_\lambda + x)^n = n(E_n + x)^{n-1} \\ &= nE_{n-1,\lambda}(x), \ (n \geq 1). \end{aligned}$$

Thus, we note that $E_{n,\lambda}(x)$ is an Apell polynomials for the pair $\left(\frac{(1+\lambda t)^{\frac{1}{\lambda}}+1}{2}, t\right)$.

From (3.1) and (3.2), we can derive

$$
\begin{aligned}
&\sum_{n=0}^{\infty} E_{n,\lambda}(x)\frac{t^n}{n!} \\
&= \sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\left(\frac{y}{\lambda}\right)^m d\mu_{-1}(y)\frac{1}{m!}\left(\log(1+\lambda t)\right)^m e^{xt} \\
&= \left(\sum_{m=0}^{\infty}\lambda^{-m}\int_{\mathbb{Z}_p} y^m d\mu_{-1}(y)\sum_{k=m}^{\infty}S_1(k,m)\frac{\lambda^k t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\frac{x^l}{l!}t^l\right) \\
&= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\lambda^{k-m}E_m S_1(k,m)\right)\frac{t^k}{k!}\times\left(\sum_{l=0}^{\infty}\frac{x^l}{l!}t^l\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\lambda^{k-m}E_m S_1(k,m)x^{n-k}\binom{n}{k}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{3.5}
$$

By comparing the coefficients on the both sides of (3.5), we get the following theorem.

Theorem 3.1. For $n \geq 0$, we have

$$
E_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}\lambda^{k-m}E_m S_1(k,m)x^{n-k}\binom{n}{k}.
$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (3.1), we get

$$
\begin{aligned}
\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y) &= \sum_{m=0}^{\infty}E_{m,\lambda}(x)\frac{\lambda^{-m}}{m!}\left(e^{\lambda t}-1\right)^m \\
&= \sum_{m=0}^{\infty}E_{m,\lambda}(x)\lambda^{-m}\sum_{n=m}^{\infty}S_2(n,m)\frac{\lambda^n t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}E_{m,\lambda}(x)\lambda^{n-m}S_2(n,m)\right)\frac{t^n}{n!},
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y) &= \frac{2}{e^t+1}e^{xt} \\
&= \sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}, \text{ (see [1-6]).}
\end{aligned}
\tag{3.7}
$$

Therefore, by (3.6) and (3.7), we obtain the following theorem.

Theorem 3.2. For $n \geq 0$, we have

$$E_n(x) = \sum_{m=0}^{n} E_{m,\lambda}(x)\lambda^{n-m} S_2(n,m).$$

As is known,

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) &= \frac{2}{e^t+1} e^{xt} \\ &= \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}.\end{aligned}$$

Thus, by replacing t by $\frac{1}{\lambda}\log(1+\lambda t)$, we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} e^{(x+y)\frac{1}{\lambda}\log(1+\lambda t)} d\mu_{-1}(y) \\ &= \sum_{m=0}^{\infty} E_m(x)\frac{1}{m!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^m \\ &= \sum_{m=0}^{\infty} E_m(x)\lambda^{-m} \sum_{n=m}^{\infty} S_1(n,m)\frac{\lambda^n t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} \lambda^{n-m} E_m(x) S_1(n,m)\right)\frac{t^n}{n!}.\end{aligned} \tag{3.8}$$

On the other hand,

$$\begin{aligned}&\int_{\mathbb{Z}_p} e^{(x+y)\frac{1}{\lambda}\log(1+\lambda t)} d\mu_{-1}(y) \\ &= \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+y}{\lambda}} d\mu_{-1}(y) = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \left(\sum_{m=0}^{\infty} E_{m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}\left(\frac{x}{\lambda}\right)_l \lambda^l \frac{t^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}(x|\lambda)_l \binom{n}{l} E_{n-l,\lambda}\right)\frac{t^n}{n!}\end{aligned} \tag{3.9}$$

where $(x|\lambda)_l = x(x-\lambda)\cdots(x-\lambda(l-1))$.

Therefore, by (3.8) and (3.9), we obtain the following theorem.

Theorem 3.3. For $n \geq 0$, we have

$$\sum_{m=0}^{n} \lambda^{n-m} E_m(x) S_1(n,m) = \sum_{l=0}^{n} (x|\lambda)_l \binom{n}{l} E_{n-l,\lambda}.$$

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