Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 11, Number 3 (2015), pp. 1577-1581

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Weak Equations In The Algebra of New Generalized Functions Space $\mathrm{A}(S(R))$

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Abstract

In this paper we will define the weak equations in the algebra of new generalized functions $\mathrm{A}(S(R))$ as well we will prove some important results and equalities which play role in applications.

Key Words: Fourier transform , new generalized function , distribution , algebra,

Introduction

In [1] we defined the algebra of new generalized functions space $\mathrm{A}(E)$ as a factor algebra $\mathrm{A}(E) = G_{\theta_1}(E) \setminus G_{\theta_2}(E)$, where E - be separated complete locally convex algebra with topology defined by a family of semi norms $(p_\alpha)_{\alpha \in I}$ such that for each $\alpha \in I$ there exists $\beta \in I$ and a constant $c_\alpha > 0$ for which

$$p_\alpha(xy) \le c_\alpha p_\beta(x) p_\beta(y) \text{ for each } x, y \in E \qquad (1)$$

As a special case we defined the algebra $\mathrm{A}(S(R))$, where $S(R)$ - be the space of functions of rapid decay with topology defined by the family of semi norms $p_\alpha(f) = p_{n,l}(f) = \sup\limits_{\substack{k \le n \\ m \le l}} q_{k,m}(f)$, where $q_{k,m}(f) = \sup\limits_{x \in R} \left| x^k f^{(m)}(x) \right|$

The topology $p_\alpha(f) = p_{n,l}(f) = \sup\limits_{\substack{k \le n \\ m \le l}} q_{k,m}(f)$ satisfies (1) [2-3].

The embedding of the space $S(R)$ and its dual space $S^*(R)$ in to the algebra $\mathrm{A}(S(R))$ defined by the following way:

If $D(R)$ the space of tests functions with compact support , then define

$M = \{\varphi \in D(R): 0 \le \varphi(x) \le 1 , \varphi(x) = 0 , |x| \ge 0 , \varphi(x) = 1 , |x| \le 1$, then define $\varphi_k = \varphi(\frac{x}{k})$, and let $g_k = F(\varphi_k(x)) = k F(\varphi(k x))$, where F is the Fourier transform .

Define the embeddings

$$J: f \in S(R) \rightarrow (f_k) + G_{\theta_2}(S(R)) \in \mathrm{A}(S(R))$$

$$J_\varphi : u \in S^*(R) \rightarrow R_\varphi = ((2\pi)^{-1} (\varphi_k . u * g_k) + G_{\theta_2}(S(R)) \in \mathrm{A}(S(R))$$

As φ runs through the set M we obtain different $R_\varphi u$ in $\mathrm{A}(S(R))$ but they are all equal in the weak sense.

On the space $\mathrm{A}(S(R))$ the linear and bilinear operations are defined [1] in particular the Extended Operations differentiation $\tilde{D}$, multiplication $\otimes$, Fourier transform $\tilde{F}$, and convolution $\tilde{*}$ are defined in the following way:

Since differentiation D and the Fourier transformation F are continuous linear operators in $S(R)$ [4] and the multiplication (.) and convolution $*$ are continuous bilinear maps on $S(R)$ [5], then they are lifted coordinate wise to operations in $\mathrm{A}(S(R))$, which we denote by $\tilde{D}, \tilde{F}, \otimes, \tilde{*}$ and defined in the following way :

$$\tilde{D} : f \in A(S(R)) \rightarrow (D(f_k)) + G_{\theta_2}(S(R));$$

$$\tilde{F} : f \in A(S(R)) \rightarrow (F(f_k)) + G_{\theta_2}(S(R));$$

$$\otimes : (f, g) \in A(S(R)) x A(S(R)) \rightarrow ((f_k . g_k)) + G_{\theta_2}(S(R));$$

$$\tilde{*} : (f, g) \in A(S(R)) x A(S(R)) \rightarrow ((f_k * g_k)) + G_{\theta_2}(S(R)).$$

These operations possess many good properties in $S(R)$ and are very convenient for applications. These properties are also preserved in the space $A(S(R))$.

Weak Equations in the Algebra $\mathrm{A}(S(R))$

Our aim is to study the relationship between the standard operations $D, F, ., *$ and their extensions $\tilde{D}, \tilde{F}, \otimes, \tilde{*}$.

Definition Let $f, g \in \mathrm{A}(S(R))$ then the elements f and g are called weakly equal $(f \approx g)$ if and only if $f_k - g_k \rightarrow 0$ in the space $S^*(R)$,where $f_k, and\ g_k$ are any representatives of $f, and\ g$ respectively .

Lemma 2.1

for each $\psi \in S(R)$ the sequence $\varphi_k \psi$ converges to ψ in $S(R)$

Proof

let $\xi(x) = \varphi(x) - 1$, and $\xi_k(x) = \varphi_k(x) - 1$. It is enough to show that $\xi_k \to 0$ in $S(R)$. Consider $\sup_{x \in R} \left| x^i (\xi_k \psi)^{(n)}(x) \right| = \sup_{|x|>k} \left| x^i \sum_{j=0}^{n} C_n^j \, \xi_k^{(j)}(x) \varphi^{(n-j)}(x) \right| \le$

$$\le 2^n \sup_{|x|>k} \left| x^i \sum_{j=0}^{n} (\frac{1}{k^j}) \xi_k^{(j)}(x) \varphi^{(n-j)}(x) \right| \le \frac{1}{k} 2^n M \sup_{|x|>k} \left| x^i \sum_{j=0}^{n} \varphi^{(n-j)}(x) \right|$$

where $M = \sup_{x \in R} \left| \xi_k^{(j)}(x) \right| \quad j = \overline{1, n}$.

Now let $f(x) = \sum_{j=0}^{n} \varphi^{(n-j)}(x) \in S(R)$, then there is a constant c such that $\left| x^{i+1} f(x) \right| \le c$, that is $\left| x^i f(x) \right| \le \frac{c}{x}$. Now $\sup_{x \in R} \left| x^i (\xi_k \psi)^{(n)}(x) \right| \le 2^n M c \frac{1}{k} \to 0$

That is $\xi_k \psi \to 0$ in $S(R)$.

Lemma 2.2

Let $v \in S^*(R)$,then the sequence $\varphi_k v$ converges to v in $S^*(R)$.

Proof

for each $\psi \in S(R)$ we have $< \varphi_k v, \psi > = < v, \varphi_k \psi > \to < v, \psi >$,

that is $\varphi_k v \to v$ in $S^*(R)$.

Theorem 2.1

For each $u \in S^*(R)$ the sequence $R_\varphi = (\ (2\pi)^{-1} (\varphi_k \, . \, u * g_k\ )$

converges to u in $S^*(R)$.

Proof

By using lemma2.1 and lemma2.2 we have $F[\varphi_k \, . \, u\ ] \to F[u]$ in $S^*(R)$, and $\varphi_k \psi \to \psi$ in $S(R)$.

Now it is known that if $f_k \to f$ in $S^*(R)$ and $\varphi_k \to \varphi$ in $S(R)$, then $< f_k, \varphi_k > = < f, \varphi >$. So $< F[\varphi_k u], \varphi_k u > \to < F[u], \psi >$ or $< F[\varphi_k u], \psi > \to < F[u], \psi >$.That is $F[\varphi_k u] \varphi_k \to Fu$ in $S^*(R)$ from which implies that $F[F[\varphi_k u] \varphi_k] \to F[F[u]]$, that is $(\ (2\pi)^{-1} (\varphi_k \, . \, u * g_k\ ) \to u$ in $S^*(R)$

Theorem 2.2

Let $u \in S^*(R)$, then a) $R_\varphi[Du] \approx D^*[R_\varphi u]$; b) $R_\varphi[Fu] \approx F^*[R_\varphi u]$

Proof

a)Let $u \in S^*(R)$,then $Du \in S^*(R)$ and from theorem 2.1 we have $R_\varphi[Du] \approx Du$ and $(\varphi_k . u * g_k) \to u$ in $S^*(R)$. That is $D^*(\varphi_k . u * g_k) \to Du$ in $S^*(R)$, then $R_\varphi[Du] \approx D^*[R_\varphi u]$.

The proof of b) is similar.

Theorem 2.3

Let $u, v \in S^*(R)$ and let $u * v \in S^*(R)$ then $R_\varphi[u * v] \approx R_\varphi[u] \otimes R_\varphi v]$

Proof

the convolution $* : S^*(R) x S^*(R) \to S^*(R)$ is bilinear continuous operator , then

$$R_\varphi[u * v] \approx u * v \tag{2.1}$$

Now let $u \in S^*(R)$ be fixed ,then $u * . : S^*(R) \to S^*(R)$ be continuous operator , that is $u * R_\varphi v \approx u * v$,but $u * R_\varphi v \approx R_\varphi[u] \otimes R_\varphi v]$,then

$$u * v \approx R_\varphi[u] \otimes R_\varphi v] \tag{2.2}$$

Now from (2.1) and (2.2) we receive $R_\varphi[u * v] \approx R_\varphi[u] \otimes R_\varphi v]$.

Acknowledgment

We would like to express our thanks to sponsor the Deanship of Scientific Research in Jazan University.

References

[1] Ramadan Sabra, Saleem Al-Ashhab , Extended Operations and New Generalized Numbers Corresponding to the Algebra A(S(R)) ,International Journal of Contemporary Mathematical Sciences, , Vol. 10, 2015, no. 4, 179-183 2015

[2] Ramadan Sabra , Mohamed Hasan. Further New Generalized Functions Spaces and Extended Linear Transformations, Journal of Nonlinear Functional Analysis . 2013. pp.(1-8).

[3] Ramadan Sabra, Ultra Linear Continuous Functionals and Ultra generalized Complex Numbers In The Ultra generalized Function Spaces,

International Journal of Open Problems in Computer Science and Mathematics . Vol.5,No.2, 2012. pp. 156-162

[4] A.B Antonevich and Ya.V. Radyno.Functional Analysis and Integral Equations. Minsk BSU 2006 ISBN 985-485-835-0.

[5] Gelfand Y. M., G. E. Shilov. The space of test and Generalized Functions Moscow. 1958.